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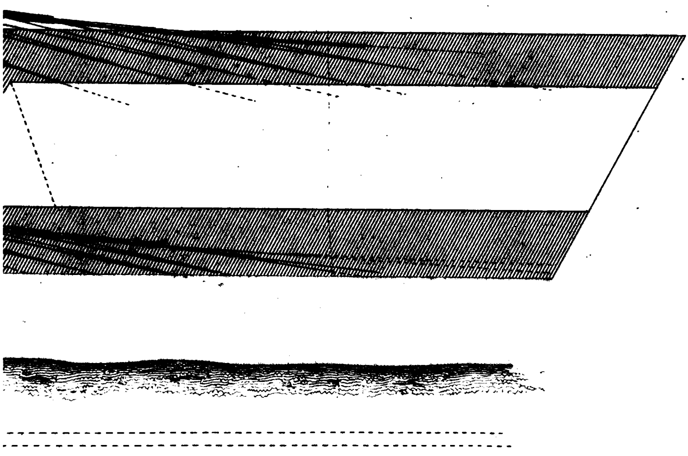

F. Mansell sc.

DREDGE'S

SUSPENSION BRIDGE

EXPLAINED UPON THE

PRINCIPLES OF THE LEVER.

BY W.illiam TURNBULL,

AUTHOR OF TREATISES ON THE STRENGTH OF CAST IRON, TIMBER, ETC. ETC.

TO WHICH ARE ADDED,

A SPECIFICATION OF THE QUANTITIES OF MATERIAL USED IN THE SUSPENSION BRIDGE AT BALLOCH FERRY, DUNBARTONSHIRE,

And an Isometrical Projection.

BY JAMES DREDGE, OF BATH.

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TO

THE RIGHT HONOURABLE LORD WESTERN.

My Lord,

In consideration of the great interest you have manifested in the success of my Invention, and the influence of your Lordship's approbation in drawing public attention thereto, the following Essay, drawn up with a design of establishing the principles mathematically, is, as a small tribute of gratitude,

Most respectfully inscribed by,

My Lord,

Your Lordship's most obedient Servant,

James Dredge.

PREFACE.

The circumstances which led to the composition of the following pages may be briefly stated, thus. Early in the spring of the present year (1841), John Macneill, Esq., an Engineer of high professional standing, proposed it to me by way of a problem:—

> *To trace the principle which induced Mr. Dredge to adopt the tapering chain and the oblique suspending rods, and to prove mathematically, that the principle thus adopted is strictly in accordance with the maxims of accurate mechanics.*

To assist in arriving at a correct conclusion, the proposer laid before me several notices that had appeared on the subject in the Mechanics' Magazine and other periodicals, and among these was a letter from Lord Western to Lord Viscount Melbourne, originally published in the *Times* newspaper, and afterwards in the Civil Engineer and Architect's Journal for June, 1840. This letter being drawn up in a strain so candid and liberal, setting forth in fervid and glowing language the confidence which the noble writer entertains of the correctness of Mr. Dredge's system, and the advantages to be derived from its general adoption,

induced me to believe that the principle, however it might have been discovered, was worthy of the most attentive consideration, and held out high claims on public patronage and support.

My attention being thus directed to the subject, I of course made it an object to obtain a solution, and for this purpose I resorted to the well known properties of the lever, supposing it to be sustained in equilibrio by a power applied at different distances from the fulcrum, and acting in given directions. In my first attempts I was not successful; for it did not then occur to me that there was a general term with which the combined effects of all the forces, whatever might be their number, ought to be compared, and this circumstance alone prevented me from obtaining a solution at the time when the problem was proposed.

On a subsequent occasion my attention was again directed to the same object, and having in the interval been shown a drawing, which was intended to illustrate the effects of the new principle as compared with those of the old, I was led to the detection of the general term on which the solution depends, and in a few days after, the theory, as discussed in the following pages, was drawn up.

The manuscript thus completed was laid before Mr. Weale, of the Architectural Library, High Holborn, who expressed a wish to purchase it forthwith, provided Mr. Dredge, who was then in London, would undertake to furnish a drawing and example illustrative of some bridge either erected or in the course of erection.

I accordingly waited on Mr. Dredge, who, in the most affable and condescending manner, accompanied me to the Publisher, and generously offered to supply whatever might be required for elucidation of the system and its application, either in the shape of drawings, specifications, or estimates; and, in accordance with this generous offer, the Contract, Specification of Materials used, and beautiful Isometric Drawing of the Bridge across the River Leven at Balloch Ferry, Dunbartonshire, which are appended to this Essay, have been supplied.

If the theory as here investigated have the effect of establishing the correctness of the principles upon which Mr. Dredge proceeds, and tend in any measure to counteract and remove the prejudices that have been entertained against its introduction by several of the most eminent and skilful Engineers, I shall consider my labours as not having been spent in vain.

W. T.

THE MATHEMATICAL PRINCIPLES

OF

DREDGE'S SUSPENSION BRIDGE.

A NOVEL and elegant principle has lately been adopted in the construction of *Suspension Bridges*,—a principle which promises to be of the greatest utility and importance, both as regards the quantity of material employed, and the stability and strength which it confers upon the structure.

One of the distinguishing features of this very ingenious invention, which is due to Mr. DREDGE, of Bath, consists in making the chains of sufficient magnitude and strength at the points of suspension, to support with safety the greatest permanent and contingent load, to which, under the circumstances of locality, they are ever likely to be exposed; and from thence, to taper or diminish them gradually to the middle of the bridge, where the strain becomes essentially evanescent.

The gradual diminution of the chains, however, is not the only circumstance which characterizes this mode of construction, and marks its superiority over the methods employed by other engineers. The suspending rods or bars that support the platform or road-way, instead of being hung vertically or

at right angles to the plane of the horizon, are inclined to it in angles which vary in magnitude from the abutments to the middle of the bridge, where the obliquity, as well as the stress upon the chains, attains its minimum value.

The principle developed by this obliquity of the suspending rods is singularly beautiful; but much judicious management is necessary on the part of the engineer, to fix upon that degree of obliquity which shall produce the greatest effect. Each bar is considered to perform its part in supporting the load, in proportion to its distance from the abutment, drawn into the sine of the angle of its direction, so that the entire series of suspending bars, transmits the same tension to the points of support as would be transmitted by a single bar reaching from thence to the middle of the bridge.

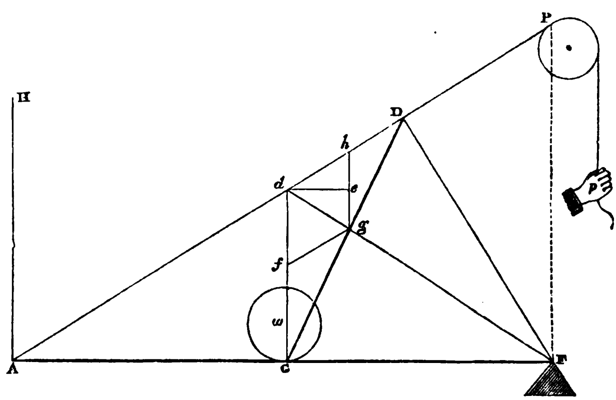
If this individual bar be considered as a straight inflexible line, the principles of calculation are identical with those of a lever when sustained in equilibrio by a single force applied at the remote extremity, and acting in the direction of the sustaining bar. But when the bar is curved or polygonal, as it necessarily must be in the case of a chain employed in the actual construction of a bridge, the process becomes a little more complicated, since it requires all the composant forces to be referred to the several portions of the chain as resultants; we mean such portions of it as are comprehended between any two contiguous bars. In every other respect, the principles which regulate the calculations are the same; and for this reason, it will be sufficient to establish the theory in reference to the straight line only, since the reader who is familiar with the *Composition* and *Resolution* of *Forces*, will find no difficulty in extending the process to a curve, and he who is not, will derive but little benefit from a perusal of the subject which is now to be laid before him.

Nothing can be more simple than the doctrine of the lever,

when sustained in equilibrio by a single force applied at a given point, and acting in a given direction: this is explained in every treatise on the principles of mechanical science, and is consequently within the reach of every individual connected with that or kindred pursuits: but a more important case of the problem remains to be considered, viz., that in which the equilibrium is maintained by a system of forces applied at different points in the length of the lever, and acting in different directions with respect to the horizon. This is the case that becomes assimilated to a suspension bridge, when the chains and suspending rods are simultaneously called into action in such a manner, that their joint effects shall be competent to maintain the system in a state of balanced rest.

In the case of a single force applied at the remote extremity of the lever, and having its direction inclined to the horizon in a given angle, while the load is applied at the middle of the length, and acting in the direction of gravity. Let A F (fig. 1)

Fig. 1.



be a lever, having one extremity resting on the fulcrum at F, and conceive it to be sustained in equilibrio by a power or force p applied at A the other extremity, and acting in the direction A P.

Conceive the lever, thus circumstanced, to be a heavy and

perfectly inflexible bar, of uniform figure and density in all its parts, and suppose the whole of its weight to be accumulated in G, the centre of gravity; it is then reduced to the case of a straight inflexible lever void of weight, and having a load of given magnitude applied at its middle point.

From the fulcrum F draw the straight line F D perpendicular to A P, the direction of the sustaining force; then, by the principles of plane trigonometry, we have the following proportion, viz.,

$$\text{rad.} : \sin. \angle \mathrm{F\,A\,D} : : \mathrm{F\,A} : \mathrm{F\,D}.$$

Or, by reducing the analogy and making radius $= 1$, it is

$$\mathrm{F\,D} = \mathrm{F\,A} \,.\, \sin. \angle \mathrm{F\,A\,D}.$$

Now, F D is the perpendicular distance between the fulcrum and the direction in which the force acts; consequently, by the property of the lever,

> *The power acting in the direction* A P, *is to the weight acting in the direction of gravity, as the distance* F G *is to the distance* F D. *That is, the power and the weight are reciprocally as the distances* F G *and* F D.

Let $p =$ the magnitude of the force which is applied at A, and acting in the direction A P,

$d_n =$ A F, the whole length of the lever, or the distance from the fulcrum of the point at which the force acts,

$\phi_n =$ F A P, the angle of direction, or that contained between the plane of the horizon and A P the direction of the force,

$w =$ the weight of the lever or platform, supposed to be concentrated at G the middle of its length,

$\delta = \frac{1}{2} d_n =$ F G, the distance between the fulcrum and the point where the weight acts.

Then, by the property of the lever just enunciated, the state of

equilibrium in terms of our notation is expressed by the following analogy, viz.,

$$p : w :: \delta : d_n \sin. \phi_n .$$

Therefore, by multiplying extremes and means, the *equation of condition* is

$$p\, d_n \sin. \phi_n = \delta\, w. \qquad \text{(A)}$$

And by division, the value of the suspending power becomes

$$p = \frac{\delta}{d_n} . w \operatorname{cosec.} \phi_n . \qquad \text{(B)}$$

Now, by the principles of mechanics, the effect must obviously be the same at whatever point in the line A P the force may be applied; so that, if A P be a cord or chain supporting the weight w by its tension, the measure of that tension is the same at every point, provided that its inclination in respect of the horizon is constantly the same. And if P be a point vertically situated with respect to the fulcrum F, let the cord or chain A P be suspended at P, in such a manner as to be kept perfectly tight by the strain which is excited in maintaining the equilibrium; then the value of the strain upon the point of suspension at P is expressed by equation (B).

Reverting to the equation of condition, and substituting $\frac{1}{2}d_n$ for δ, the expression for the state of equilibrium becomes

$$p \sin. \phi_n = \tfrac{1}{2} w, \qquad \text{(C)}$$

or, by division, we obtain for the equilibrating power,

$$p = \tfrac{1}{2} w \operatorname{cosec.} \phi_n . \qquad \text{(D)}$$

This expression for the measure of the power which maintains the system in a state of rest is exceedingly simple; but being limited to the case where the weight acts at the middle of the lever, and the sustaining power at the remote extremity, it will be more advantageous in our subsequent inquiries to retain

the general value, as expressed above, in terms of the respective distances d_n and δ; because, in as far as the lever is concerned, the symbol δ may partake of all values less than d_n, the entire length of the bar on which the forces act.

If the power p, instead of acting in the direction A P oblique to the horizon, should act in the direction A H perpendicular to it, then $<$ F A H $=\phi_n=90^\circ$, and cosec. $\phi_n=1$; in which case we have

$$p=\tfrac{1}{2}w. \tag{E}$$

This is manifest; for one half the weight is sustained by the fulcrum, and the other half by the power applied at A.

In what precedes, we have established the equation of condition by reference to the principles of the lever only; but the same thing can readily be accomplished by the *Resolution* of *Forces*, and by the same means we at once obtain the magnitude of the oblique strain upon the fulcrum or abutment, and also the horizontal thrust. Through the point G (fig. 1) draw the straight line G d at right angles to A F, and meeting A P the direction of the power in d. Draw d F and D G, and from the point d on the vertical line d G set off df at pleasure, to denote the magnitude of the weight w acting in the direction of gravity. Upon df complete the parallelogram $dfgh$, and fg or dh will represent the magnitude of the force in the direction A P, while the diagonal dg denotes the oblique pressure on the fulcrum, or the magnitude of the force in the direction d F.

Since the angles A D F and A G d are right angles by construction, the quadrilateral figure F D d G can be inscribed in a circle, and because the angles F D G and F d G are subtended by the same chord F G, they are equal between themselves; the angles D F G and dfg are also equal, each of them being equal to the complement of the given angle P A F. Con-

sequently, the triangles F D G and fdg are similar, and their homologous or corresponding sides proportional; so that

$$FD : fd :: FG : fg.$$

Now F D, as we have already seen, is equal to F A . sin. $<$ F A D $= \underset{n}{d}$ sin. $\underset{n}{\phi}$, and $fd = w$, F G and fg being respectively represented by the symbols δ and p; hence we get

$$\underset{n}{d} \sin. \underset{n}{\phi} : w :: \delta : p,$$

from which, by equating the products of the extremes and means, we obtain

$$p \underset{n}{d} \sin. \underset{n}{\phi} = \delta w . \text{ (the same as equation A).}$$

In the plane triangle dfg we have given the two sides df and fg, with the contained angle dfg, to find the third side dg; and for this purpose, a well known theorem in plane trigonometry gives

$$dg = \left\{ w^2 + p^2 - 2\, wp \sin. \underset{n}{\phi} \right\}^{\frac{1}{2}} \qquad \text{(F)}$$

But, according to equation (B), the value of p in terms of $\underset{n}{d}$, δ, w and $\underset{n}{\phi}$ is $\frac{\delta}{\underset{n}{d}} . w \operatorname{cosec.} \underset{n}{\phi}$; and consequently, its square is $\frac{\delta^2}{\underset{n}{d^2}} . w^2 \operatorname{cosec.}^2 \underset{n}{\phi}$; therefore, by substituting these values for p and p^2 within the brackets, we get

$$dg = w \sqrt{1 + \frac{\delta^2}{\underset{n}{d^2}} \operatorname{cosec.}^2 \underset{n}{\phi} - \frac{2\,\delta}{\underset{n}{d}} \operatorname{cosec.} \underset{n}{\phi} \sin. \underset{n}{\phi}} .$$

By the principles of trigonometry, the cosecant of any arc or angle to radius unity, is the same as the reciprocal of the sine to the same radius; hence we have cosec. $\underset{n}{\phi}$ sin. $\underset{n}{\phi} = 1$, so that the expression for the oblique thrust becomes

$$r = \frac{w}{\underset{n}{d}} \sqrt{\underset{n}{d^2} - 2\, \underset{n}{d}\, \delta + \delta^2 \operatorname{cosec.}^2 \underset{n}{\phi}} . \qquad \text{(G)}$$

Where the symbol r denotes the resultant of the weight or load w and the oblique force p, and corresponds in magnitude and direction to the oblique thrust on the fulcrum at F.

There is still another quantity remaining to be determined, the magnitude of which may sometimes be required in a case of actual construction; we mean the horizontal thrust upon the abutment, occasioned by the joint effects of the weight and the oblique power, when exerting themselves with full intensity in their respective directions.

Let dg (fig. 1), which represents the oblique thrust upon the abutment, be resolved into the two forces de and ge, the one of them being parallel to the horizon, and the other perpendicular to it; then it is obvious that in so far as the thrust upon the abutment is concerned, the effect of the vertical force is wholly lost, that of the horizontal force only being employed in producing the strain.

Since, by construction, the straight line de is parallel to A F, the angle hde is equal to the angle $\mathrm{PAF} = \phi_n$, and we have already seen, equation (B), that fg or its equal dh is expressed by $\frac{\delta}{d_n} . w \operatorname{cosec.} \phi_n$; therefore by the principles of plane trigonometry we get

$$\text{rad.} : \frac{\delta}{d_n} . w \operatorname{cosec.} \phi_n : : \cos. \phi_n : de.$$

Therefore, if h be put to denote the horizontal thrust upon the abutment, the reduction of the above proportion will give

$$h = \frac{\delta}{d_n} . w \operatorname{cosec.} \phi_n \cos. \phi_n .$$

But, according to the calculus of sines, the product of the cosine and cosecant of any arc or angle is equal to the cotangent; hence we have $\operatorname{cosec.} \phi_n \cos. \phi_n = \cot. \phi_n$, from which, by substitution, we get

$$h = \frac{\delta}{d_n} . w \cot. \phi_n . \qquad \text{(H)}$$

It therefore appears that the equations (B), (G), and (H), exhibit the several particulars required in and arising from the equilibration of a lever; and in order that these particulars may be clearly understood, we shall now proceed to show in what manner the equations are to be reduced, by applying them to the resolution of a numerical example.

EXAMPLE.—Suppose the length of a lever to be 56 feet, each foot in length being equal to a weight of 40·56 lbs. avoirdupoise; now, if one extremity of this lever rests upon a fulcrum, while it is sustained in equilibrio by a force applied at the other extremity, and acting in a direction which makes an angle of 32 degrees with the horizon; what is the magnitude of the sustaining force, and what are the thrusts excited upon the fulcrum, both in reference to the oblique and horizontal direction?

The first demand of this question is fulfilled by equation (B), the second by (G), and the third by (H); in all of which, the value of δ is 28 feet, being equal to one half the length of the lever. And since the system is placed under the same conditions as it would be, if a load equivalent to the weight of the lever were applied at its centre of gravity, the lever itself being then considered as without weight, it follows that

$$w = 56 \times 40{\cdot}56 = 2271{\cdot}36 \text{ lbs.}$$

Since the angle of direction of the suspending power is 32 degrees, its natural cosecant is 1·88708, and the numerical expression for the magnitude of the power by which the equilibrium is maintained is

$$p = \frac{28 \times 2271{\cdot}36 \times 1{\cdot}88708}{56} = 2143{\cdot}12 \text{ lbs.}$$

The logarithmic process for the reduction of this expression is as follows, viz.,

$$\left.\begin{array}{lr} \delta = 28 \text{ feet} \dots\dots\dots & \text{log. } 1{\cdot}4471580 \\ w = 2271{\cdot}36 \text{ lbs.} \dots\dots\dots & \text{log. } 3{\cdot}3562859 \\ \phi_{\text{s}} = 32 \text{ degrees} \dots\dots & \text{log. cosec. } 0{\cdot}2757903 \\ d_{\text{s}} = 56 \text{ feet} \dots\dots\dots & \text{ar. co. log. } 8{\cdot}2518120 \end{array}\right\} \text{add.}$$

Therefore the value of p, the equilibrating power,
is 2143·12 lbs. . . . . log. 3·3310462 sum.

Hence it appears, that a power of 2143·12 lbs., acting at the remote extremity of the lever, and having its direction inclined to the horizon in an angle of 32 degrees, will balance a load of 2271·36 lbs. applied at the middle of the length, and acting in the direction of gravity, the lever itself having no weight.

The data remaining as above, the numerical expression for the value of the oblique thrust on the abutment is

$$r = \frac{2271{\cdot}36}{56}\left\{56^2 - 2 \times 56 \times 28 + 28^2 \times 1{\cdot}88708^2\right\}^{\frac{1}{2}}$$

By a slight examination of the quantities within the brackets, it will readily appear that the second or negative term is precisely equal to that which precedes it; consequently, they destroy one another, and the expression reduces to the following, viz.,

$$r = \frac{2271{\cdot}36 \times 28 \times 1{\cdot}88708}{56} = 2143{\cdot}12 \text{ lbs.}$$

From this we infer that when the load is applied at the middle of the lever, and the balancing power at the remote extremity, the magnitude of the oblique thrust on the fulcrum is precisely the same as the magnitude of the sustaining power. This is also manifest from the figure; for the triangle $g\,d\,h$ or $d\,g\,f$ is isosceles.

Finally, the data still remaining, the numerical expression for the magnitude of the horizontal thrust on the fulcrum becomes

$$h = \frac{28 \times 2271{\cdot}36 \times 1{\cdot}6003345}{56} = 1817{\cdot}47 \text{ lbs.}$$

Where 1·6003345 is the natural cotangent of 32 degrees; hence the following logarithmic process,

$\frac{\delta}{d}=\frac{28}{56}$ . . . . log. 9·6989700

$w = 2271·36$ . . . . log. 3·3562859

$\phi = 32°$ . . . . log. cot. 0·2042108

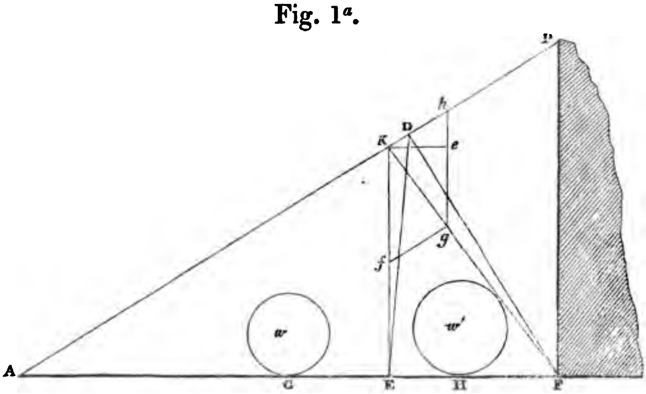
Therefore the value of *h*, the horizontal thrust, is

1817·47 lbs. . . . . log. 3·2594667 sum.

The preceding investigation has reference only to the weight of the lever or platform, which is constantly the same; but in the case of a bridge, this constancy of weight cannot obtain by reason of the transit loads to which it is exposed; and as the loads of transit are continually varying their distances from the fulcrum, and thereby producing different effects at every instant, it becomes necessary, in order to render our investigation complete, to show in what manner these effects are to be estimated, and incorporated with the effect produced by the weight of the platform.

Let w (fig. 1[a]), denote the weight equivalent to that of the

Fig. 1[a].



platform, applied at G the centre of gravity or middle point of the lever, and let w' be any other load applied at the point H.

Then it is manifest, from the principles of the lever, that the two weights w and w' will, according to their situations, perform the same office in equipoising the system, as would be performed by a single weight equivalent to their sum applied at E their common centre of gravity, and acting in a direction perpendicular to the horizon. Divide the distance G H into two parts G E and H E, which are to one another reciprocally as the weights applied at G and H; then is E the position of the load, which is equivalent to $w + w'$. Through the point E, and perpendicular to F A, draw the straight line E K, meeting A P in K; and let F D be perpendicular to A P. Draw D E and F K, and upon K E set off K f, equal to the compound load $w + w'$, and complete the parallelogram K $f\,g\,h$; then will $f\,g$ or K h denote the force in direction A P, and K g the oblique pressure on the fulcrum at F, arising from the conjoint action of the constant weight w and the extraneous weight w'. Through the point K draw K e at right angles to $h\,g$, and K e will denote the horizontal thrust upon the fulcrum, produced by the conjoint action of the two forces already specified; and if the load w' should move forward or backward to any other point of the lever, the same mode of construction will still apply, so that the principle is general, whether the load be constant or variable.

If the symbol δ denote the distance F E between the fulcrum F and the common centre of gravity of the two weights w and w', the other quantities remaining as before, then the equation of *condition* for the constant and variable load becomes

$$p\, d_n \sin.\, \phi_n = \delta\, (w + w'), \qquad (\mathrm{H}^a)$$

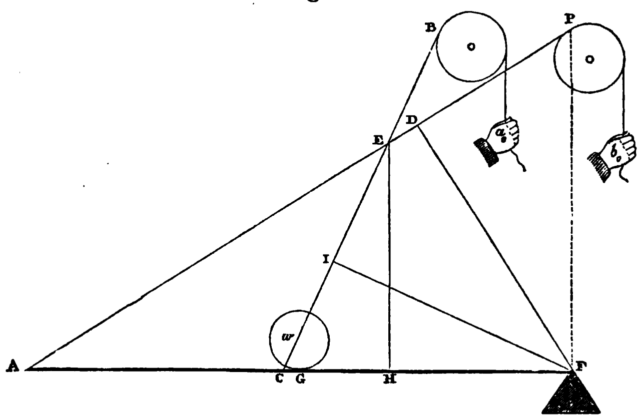
and the absolute value of the force in direction A P is

$$p = \frac{\delta}{d_n} . \, (w + w')\, \mathrm{cosec.}\, \phi_n. \qquad (\mathrm{H}^b)$$

In the next place, instead of the lever being sustained in

equilibrio by a single power p, applied at A and acting in the direction A P (fig. 2), let us suppose that the equilibrium is

Fig. 2.



produced by the joint effects of two forces b_0 and a_0, the one being applied at A the extremity of the lever, and acting in the direction A P, while the other is applied at some intermediate point of the lever, as C, and acting in the direction C B, the angles of inclination P A F and B C F being respectively denoted by the symbols ϕ_n and ϕ_0, while the distances A F and C F are indicated by d_n and d_0, the symbol δ being the same as in the preceding case; namely, the distance between the fulcrum and the point where the weight acts; that is, the distance G F.

Let the point C, at which the subsidiary force a_0 is applied, lie between the points A and G; that is, between the middle of the lever and its extreme point at A; then, since the magnitude of the weight w and its distance from the fulcrum are the same as before, its effect upon the lever will manifestly be the same also, and consequently the joint effects of the two forces b_0 and a_0, which produce the equilibrium in this case, will

be equivalent to the effect of the single force p, by which it was previously produced.

But by the laws of mechanics the effect of any force which acts upon a lever at a given distance from the fulcrum, and in a given direction,—

> *Is expressed by the magnitude of the given force, drawn into the distance between the fulcrum and the point at which it acts, and again into the sine of the angle of its direction.*

We have already seen, equation (A), that the effect of a single force which balances the weight of the lever is expressed by $p d_n \sin. \phi_n$; this, therefore, is the quantity to which the effects of the forces b_0 and a_0, now balancing the weight, must be equal, and with which they must jointly be compared.

Now, by the property of the lever as above enunciated, the effect of the force b_0 applied at A, and acting in the direction A P, is $b_0 d_n \sin. \phi_n$, the distance A F and angle P A F being the same as before. And in like manner the effect of the force a_0 applied at the point C, and acting in the direction C B, is $a_0 d_0 \sin. \phi_0$; consequently, by taking the sum of these effects, the equation of condition or equilibrium in the case of two sustaining forces becomes

$$a_0 d_0 \sin. \phi_0 + b_0 d_n \sin. \phi_n = p d_n \sin. \phi_n.$$

But by equation (A) we have $p d_n \sin. \phi_n = \delta w$; consequently, by substitution, we obtain

$$a_0 d_0 \sin. \phi_0 + b_0 d_n \sin. \phi_n = \delta w. \qquad \text{(I)}$$

The composition of this equation is very simple; for from the fulcrum F draw the straight lines F D and F I respectively perpendicular to A P and C B, the direction in which the forces

act. Then, since $A F = d_n$ and $C F = d_0$, while the angles $P A F$ and $B C F$ are respectively denoted by the symbols ϕ_n and ϕ_0, it follows from the principles of plane trigonometry, that

$$F D = d_n \sin. \phi_n , \; F I = d_0 \sin. \phi_0 .$$

And multiplying these distances by the magnitudes of the respective powers, the products will indicate the mechanical effects, and the sum of those effects must be equal to the magnitude of the weight w drawn into the distance $F G$; and this equality constitutes the equation of *condition* or *equilibrium* given above.

If we transpose the first term of equation (I), the effect of the force b_0 becomes

$$b_0 d_n \sin. \phi_n = \delta w - a_0 d_0 \sin. \phi_0 ,$$

and by division we obtain

$$b_0 = \frac{\delta w - a_0 d_0 \sin. \phi_0 ,}{d_n \sin. \phi_n}.$$

But, according to the calculus of sines, the sine and cosecant of any arc or angle to radius unity are reciprocally equal; consequently, a more elegant expression for the magnitude of b_0 is that which follows, viz.,

$$b_0 = \frac{\left(\delta w - a_0 d_0 \sin. \phi_0\right) \operatorname{cosec.} \phi_n}{d_n}. \qquad \text{(K)}$$

This supposes the magnitude of the force a_0, as well as all the other quantities composing the equation except b_0, to be known; but if b_0 be the given force, the equation expressing the value of a_0 becomes

$$a_0 = \frac{\left(\delta w - b_0 d_n \sin. \phi_n\right) \operatorname{cosec.} \phi_0}{d_0}. \qquad \text{(L)}$$

If the equations (B) and (K) be compared with each other, it will readily appear that the former exceeds the latter by the quantity $\frac{a_0 d_0 \sin. \phi_0}{d_n} \times \text{cosec.}\ \phi_n$; for it is obvious that

$$p - b_0 = \frac{\delta}{d_n} . w \text{ cosec.}\ \phi_n - \frac{\left(\delta w - a_0 d_0 \sin. \phi_0\right) \text{cosec.}\ \phi_n}{d_n} = \frac{a_0 d_0 \sin. \phi_0 \text{ cosec.}\ \phi_n}{d_n}.$$

We therefore infer, that in the case of a single subsidiary force a_0, as in fig. 2, the tension upon the cord or chain A P is less than the tension in fig. 1, by the quantity

$$\frac{a_0 d_0 \sin. \phi_0 \text{ cosec.}\ \phi_n}{d_n},$$

so that the diameter or section of the cord or chain may be diminished in the same proportion, the force in the direction C B diminishing the strain to that extent.

If the force a_0, which is applied at the point C, have its direction perpendicular to the length of the lever A F; then $\sin. \phi_0 = 1$, and the effect in that case is simply $a_0 d_0$ instead of $a_0 d_0 \sin. \phi_0$, and the corresponding value of b_0 is therefore

$$b_0 = \frac{\left(\delta w - a_0 d_0\right) \text{cosec.}\ \phi_n}{d_n}. \qquad \text{(M)}$$

If the equations (B) and (M) be now compared with each other, it will be seen that the former exceeds the latter by the quantity $\frac{a_0 d_0 \text{ cosec.}\ \phi_n}{d_n}$; for, by subtraction, it is

$$p - b_0 = \frac{\delta}{d_n} . w \text{ cosec.}\ \phi_n - \frac{\left(\delta w - a_0 d_0\right) \text{cosec.}\ \phi_n}{d_n} = \frac{a_0 d_0 \text{ cosec.}\ \phi_n}{d_n}.$$

Now, when the angle of inclination ϕ_0 is of any magnitude

whatever less than 90 degrees or a right angle, sin. ϕ_0 is less than unity, and, consequently, this last remainder is greater than the former by the quantity $\frac{a_0 d_0 \text{cosec.} \phi_n}{d_n}(1 - \sin. \phi_0)$; for by subtraction we get

$$\frac{a_0 d_0 \text{cosec.} \phi_n}{d_n} - \frac{a_0 d_0 \sin. \phi_0 \text{cosec.} \phi_n}{d_n} = \frac{a_0 d_0 \text{cosec.} \phi_n}{d_n}(1 - \sin. \phi_0).$$

From this we infer that when the direction of the force a_0 is perpendicular to the lever, the magnitude of the force b_0 in direction A P necessary to maintain the equilibrium, is less than when the direction of the force a_e is oblique. This inference, at first sight, would seem to be at variance with the principles insisted on by Mr. Dredge; and it manifestly is so, if we conceive the vertical and oblique forces to be applied at the same point of the lever; but if we suppose them to be referred to the same point of the cord or chain A P, the case will be very different; for then the quantity indicated by the symbol d_0 diminishes, and the quantity of diminution is proportional to the cosine of the inclination or obliquity. For, draw the perpendicular E H, then C F is the length of the lever in the case of the oblique force in direction C E, and H F its length for the vertical force in direction H E; but C H = C F − H F, and C H = C E cos. ϕ_e; for, by the principles of plane trigonometry, it is

$$\text{rad.} : \text{C E} :: \cos. \phi_e : \text{C H} = \text{C E} \cos. \phi_e.$$

In this equation, however, the quantity C E is still unknown, but it may be determined in the following manner:

$$\text{A C} = \text{A F} - \text{C F} = d_n - d_e,$$

and by the property of the plane triangle, as demonstrated in the 32nd proposition of the first book of Euclid's Elements of Geometry, the angle E C F is equal to the sum of the angles E A C and A E C; therefore by transposition it is

$$\text{A E C} = \phi_0 - \phi_n,$$

and by trigonometry we have

$$\sin.\left(\phi_0 - \phi_n\right) : d_n - d_0 :: \sin.\,\phi_n : \text{C E},$$

and by reducing the analogy it becomes

$$\text{C E} = \frac{\left(d_n - d_0\right)\sin.\,\phi_n}{\sin.\left(\phi_0 - \phi_n\right)}. \qquad \text{(N)}$$

Let this value of C E be substituted instead of it in the above value of C H, and we obtain

$$\text{C H} = \frac{\left(d_n - d_0\right)\sin.\,\phi_n\cos.\,\phi_0}{\sin.\left(\phi_0 - \phi_n\right)}. \qquad \text{(O)}$$

By a well known theorem in the calculus of sines, we have

$$\sin.\left(\phi_0 - \phi_n\right) = \sin.\,\phi_0\cos.\,\phi_n - \cos.\,\phi_0\sin.\,\phi_n;$$

therefore, by substitution, we get

$$\text{C H} = \frac{\left(d_n - d_0\right)\sin.\,\phi_n\cos.\,\phi_0}{\sin.\,\phi_0\cos.\,\phi_n - \cos.\,\phi_0\sin.\,\phi_n},$$

and by expunging the factor $\sin.\,\phi_n\cos.\,\phi_0$ from both terms of the fraction, it finally becomes

$$\text{C H} = \frac{d_n - d_0}{\tan.\,\phi_0\cot.\,\phi_n - 1}. \qquad \text{(P)}$$

If from C F the leverage for the oblique force in direction C B, we subtract the value of C H, as found above, the remainder H F will be the leverage for the vertical force in direction H E. Thus we have

$$H F = \frac{d_0 \tan. \phi_0 \cot. \phi_n - d_n}{\tan. \phi_0 \cot. \phi_n - 1}. \qquad (Q)$$

When the value of the distance H F, as found above, is less, equal to, or greater than F I $= d_0 \sin. \phi_0$, the effect of the vertical force will be less, equal to, or greater than that of the oblique force; by this we mean, that the magnitude of the force being the same in both cases, its effect in the vertical direction will be less, equal to, or greater than its effect in the oblique direction, according as H F is less, equal to, or greater than F I.

Since E is a fixed point through which the directions of both the vertical and oblique forces must pass, it follows that while the distance F H is constant, the distance F C may partake of all magnitudes between F A and F H, while the angle E C F may librate between the angles F H E and F A P; that is, it may vary from a right angle to ϕ_n, which is its minimum limit.

From this view of the subject it is obvious that the perpendicular F I depends upon the magnitude of the angle E C F; and because the point C may move either towards A or H, while the straight line C B passes constantly through E, it follows, that the inclination admits of such a magnitude as to give the force in the direction C B a maximum effect; that is, when $d_0 \sin. \phi_0$ is a maximum or the greatest possible.

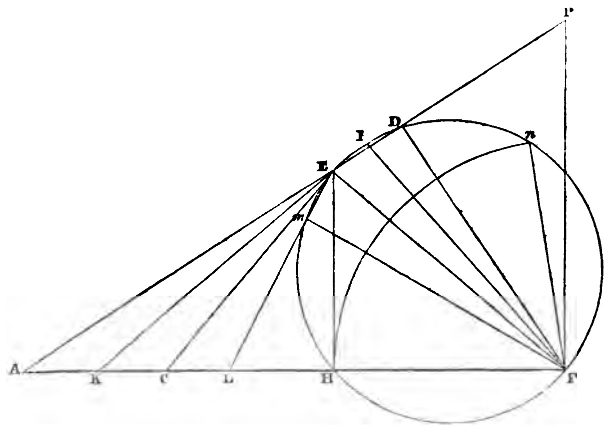
The method of determining the angle which gives to the oblique force its maximum effect is very simple; but if such a principle were adopted in the construction of suspension bridges, the results in many instances would differ very widely from the practice of the inventor, and especially as applied to the suspending rods in the *Victoria Bridge* across the Avon at Bath.

From drawings which we have seen of that structure, it would appear that the angles of direction diminish according to some law, as we recede from the abutments towards the centre

of the bridge; but in making the obliquity such as to give the force in any direction its maximum effect, the angles continually increase instead of decreasing, as will become manifest from the following constructions.

Let A F (fig. 3) be a lever resting on the fulcrum at F, and

Fig. 3.



sustained in equilibrio by a cord or chain applied at A, and fastened to a hook or some other immoveable object at P.

Let E be any point in the length of the cord, at which a subsidiary force is to be applied in order to lessen the strain upon A E, the lower portion thereof. The point C is required, so that C E being joined, a given force acting in the direction C E may produce a maximum effect.

Draw the straight line F E to connect the fulcrum with the given point, and through the point E, draw E C at right angles to F E, and meeting the lever A F in C; then is C E the direction of the force required. Upon F E, as a diameter, describe the circle F H E, intersecting the lever in the point H, and draw E H; then is E H perpendicular to the lever in H, so that H E would be the direction of the vertical force passing through E the given point.

If the point C be supposed to move towards A in such a

manner that C E may become K E; then it is manifest that K E produced will intersect the circumference of the circle again in I; draw F I, and F I will be perpendicular to K E, the direction of the given force; this is obvious, for the angle in a semicircle is a right angle. Now F I is manifestly less than F E, for F E is the diameter of the circle F H E, and by the property of the circle, the diameter is the greatest chord that can be inscribed in it. When the point K coincides with A, the point I coincides with D, in which case $F D = d_n \sin. \phi_n$.

Again, suppose the point C to move towards H until it arrives at the point L; then L E being drawn will necessarily intersect the circumference of the circle in the point *m*. Draw F *m*, which, by the property of the circle, will be perpendicular to L E, the direction of the given force supposed to be applied at L. Now, since F E is the diameter of the circle F H E, and F *m* any chord in it not passing through the centre, F E is greater than F *m*; but it has also been shown to be greater than F I; and since the effect of the force in the directions K E, C E, and L E, is represented by the perpendiculars F I, F E, and F *m*, it follows, that the effect in the direction C E is a maximum; hence the truth of the construction is manifest. When the point L coincides with H, the straight line L E coincides with H E, and H E becomes the direction of the vertical force, the effect of which is represented by F H.

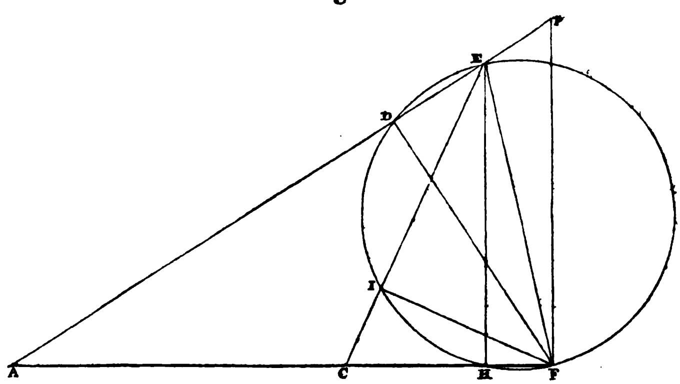
Since H and D are the points in which the circle is intersected by the lever A F, and the cord or chain A P, it follows, that the effect of the given force may be represented by all the chords that can be drawn in the circle between the points D and H, F D being the minimum value when the point C moves towards A, and F H the minimum value when it moves in the opposite direction. And all these values will manifestly be greater than F H; for from the centre F with the distance F H, describe the arc H *n* intersecting the circumference of the

circle in *n*; then it is evident that all the chords originating at F, and meeting the circumference between the points *n* and H, must be greater than F *n* or F H. Thus far, therefore, there is a manifest advantage in causing the force to act in the oblique direction.

The preceding construction determines the point in the lever at which the oblique force must be applied to produce a maximum effect for all points in the cord or chain between A and D; but it does not apply to points on the opposite side of D towards P, for in that case the direction of the force would meet the lever on the opposite side of A, which is manifestly contrary to the conditions of the problem as applied to the construction of bridges; for all the suspending bars on one side are supposed to meet the platform between the abutment and the middle of the bridge. The fact is, the direction which gives the maximum value is the direction of the cord itself, the force being applied at the point A; this will be obvious from the subjoined construction.

Let A F (fig. 4) be the lever resting upon the fulcrum at F,

Fig. 4.



and A P the direction of the force, which, acting singly at A, maintains the equilibrium. Let C be the point at which the subsidiary force is applied, and C E the direction in which it

acts; then, *it is proposed to show, that the greatest effect will be produced by the given power when it acts in the direction* A E *coincident with the cord or chain* A P.

Draw the straight line F E, connecting the fulcrum with that point in the cord through which the direction of the force passes, and upon F E as a diameter describe the circle F H D E, intersecting the lever A F in the points F and H; the cord A P in the points D and E, and C E the direction of the force in the point I. Draw F I, F D, and E H, these will respectively be perpendicular to the lines C E, A P, and A F. Now, it is obvious, that if the point C moves along the lever A F, either towards A or H, the straight line C E will intersect the circle in points towards D or H accordingly, and the perpendicular F I will increase or diminish according as the intersections fall towards D or H.

When the point C coincides with A the extremity of the lever, the straight line C E will coincide with A E, the direction of the cord, and the perpendicular F D will then attain its maximum value; hence the truth of the proposition is manifest.

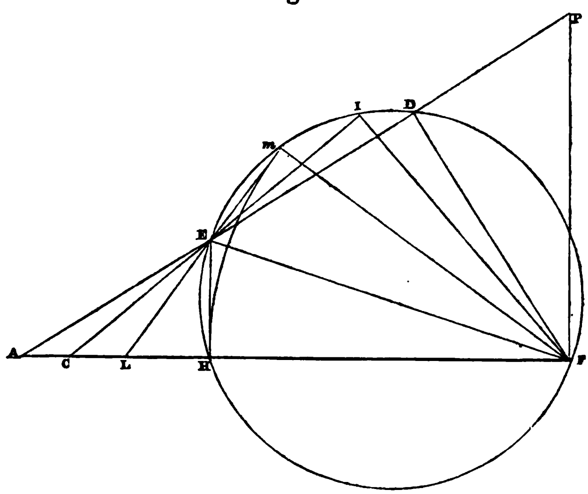
Again, when the point C coincides with the point H, the straight line C E will coincide with the perpendicular H E; and in that case the value of F H is a minimum; from which we infer that in all cases where the direction of the subsidiary force intersects the cord between the points D and P, the effect is the least possible when the force is exerted in a vertical direction.

In both the preceding constructions the position of the point E has been so assumed as to give the force, when acting in an oblique direction, a greater effect than when it acts in a vertical direction; and lest the inattentive reader should imagine that this is universally the case, we shall, in what immediately follows, endeavour to show that the effect of the

force acting vertically, may also be either equal to or greater than the effect of the same force acting obliquely, when the directions in both cases pass through the same point of the cord.

Let A F (fig. 5) be the lever, similarly situated as in all

Fig. 5.



the preceding cases, A P being the direction of the single equilibrating power. Take any point E in A P, and let the point E thus assumed be that through which the direction of the subsidiary force has to pass.

Draw F E, and upon F E as a diameter describe the circle F H E D, intersecting the lever in F and H, and the cord or chain A P in E and D. Let C be any point in the lever at which the subsidiary force is applied, and through the point E draw the straight line C E, meeting the circumference of the circle in I; then is C E the direction of the force, and F I perpendicular to C I, the representative of its effect. About the fulcrum F as a centre, with the distance F H describe the arc H *m* intersecting the circumference of the circle in *m*, and draw the straight lines F *m* and *m* E L; then is L E the direction of the oblique force acting at the point L, and the perpendicular

F m is the representative of its effect. But F m is equal to F H by the construction, and F H denotes the effect of the vertical force applied at H, and having its direction passing through E, the given point; therefore, in this case, the effects of the vertical and oblique forces are the same.

Now, it is manifest from the figure that the chord F m is greater than any other chords that can be drawn from the fulcrum F to terminate in the arc m D; therefore F m is greater than F I; but F I is the representative of the effect produced by the oblique force acting at C in the direction C E, and F m, as we have just shown, denotes the effect of the vertical force acting at H in the direction H E; so that the effect of the oblique force may be either greater, equal to, or less than that of a vertical force of the same magnitude, supposing the directions in which they act to pass through the same point.

Having thus demonstrated the conditions that must be satisfied in order to produce a maximum effect, and having shown that the value of the oblique force may be either equal to, greater, or less than that of the vertical force whose direction passes through the same point of the cord, it now remains for us to show in what manner the several equations which we have investigated are to be reduced.

Example.—Let the length of the lever, its weight, and the value of the angle ϕ_n remain, as in the example under equation (H), and suppose a subsidiary force a_0 to be applied at C at the distance of 26·5 feet from A the remote extremity of the lever, and having its direction passing through E at a distance of 44 feet from the point A (see fig. 2 preceding). It is required to determine the value of the force b_0, or the tension upon the lower portion of the cord A E, on the supposition that the required force acting in direction A E, produces the same effect in maintaining the equilibrium, as the force a_0 acting in direction C E.

The first step to be performed in the resolution of this example is to determine the angle F C E $= \phi_0$; and for this purpose we have given the length of the lever F A = 56 feet, and the angle F A P = 32 degrees; therefore, by the rules of plane trigonometry, we have

$$A\,P = 56 \text{ sec. } 32^\circ, \text{ and } F\,P = 56 \text{ tan. } 32^\circ.$$

These lines, however, are not necessary in the calculation; because, from the right-angled triangle A H E, the quantities A H and H E can be found directly from the data, without having recourse to similar triangles: thus we get

$$A\,H = 44 \text{ cos. } 32^\circ, \text{ and } H\,E = 44 \text{ sin. } 32^\circ;$$

the absolute numerical values being $44 \times \cdot 8480481 = 37\cdot 3141164$, and $44 \times \cdot 5299193 = 23\cdot 3164492$ feet respectively. Therefore, by subtraction, it is C H = $37\cdot 3141164 - 26\cdot 5 = 10\cdot 8141164$ feet; consequently, by trigonometry, we have

$$\frac{H\,E}{C\,H} = \frac{23\cdot 3164492}{10\cdot 8141164} = 2\cdot 1561122 = \text{tan. } \phi_0 = \text{tan. } 65^\circ\ 7'\ 5''.$$

Having thus determined the angle of direction for the subsidiary force a_0, the equations by which the actual values of a_0 and b_0 are to be found are the following, viz.,

$$a_0\,d_0 \text{ sin. } \phi_0 = b_0\,d_n \text{ sin. } \phi_n,$$

$$\text{and } a_0\,d_0 \text{ sin. } \phi_0 + b_0\,d_n \text{ sin. } \phi_n = p_n\,d_n \text{ sin. } \phi_n = \delta\,w,$$

or, by substituting in this last equation the effect of the force a_0 in the first, we get

$$2\,b_0\,d_n \text{ sin. } \phi_n = \delta\,w.$$

Therefore, by restoring the numerical values of the several given quantities, the corresponding value of b_0 becomes

$$b_0 = \frac{28 \times 2271\cdot 36}{2 \times 56 \times \cdot 5299193} = \frac{567\cdot 84}{\cdot 5299193} = 1071\cdot 56 \text{ lbs.}$$

Here, then, we have a strain of 1071·56 lbs. upon the lower

part of the cord; but by the example under equation (H), we found the strain throughout the cord with only one equilibrating force to be $p=2143{\cdot}12$ lbs., and this corresponds to the tension on the upper part in the present instance; hence it appears that by the application of the oblique force a_0, the lower part of the cord is relieved of a strain equal to $2143{\cdot}12 - 1071{\cdot}56 = 1071{\cdot}56$ lbs., being just half the original tension; and, consequently, the strength of the cord may be diminished in the same proportion. To find the magnitude of the force a_0 which acts in the direction C E, we must have recourse to the equation

$$a_0 d_0 \sin. \phi_0 = b_0 d_n \sin. \phi_n.$$

Now d_0, the distance between the fulcrum and the point in the lever where the force a_0 acts, is $d_0 = \text{A F} - \text{A C} = \text{C F} = 56 - 26{\cdot}5 = 29{\cdot}5$ feet; therefore, by substituting the several numerical values in the above equation, the value of a_0 becomes

$$a_0 = \frac{1071{\cdot}56 \times 56 \times {\cdot}5299193}{29{\cdot}5 \times {\cdot}9071767} = 1188{\cdot}23 \text{ lbs. nearly.}$$

Suppose, now, that the magnitude of the force a_0 and the angle of its direction are actually given by the question, instead of being determined by calculation as above, then the value of the force b_0 will be found as follows from equation (K), where we have

$$b_0 = \frac{(28 \times 2271{\cdot}36 - 1188{\cdot}23 \times 29{\cdot}5 \times {\cdot}9071767) \times 1{\cdot}8870799}{56} = 1071{\cdot}56 \text{ lbs.}$$

If the direction of the force a_0 forms with the lever an angle of 90°, then by equation (M) the value of b_0 becomes

$$b_0 = \frac{(28 \times 2271{\cdot}36 - 1188{\cdot}23 \times 18{\cdot}6858836) \times 1{\cdot}8870799}{56} = 1394{\cdot}92 \text{ lbs.}$$

The difference between the two values of b_0 found above is $1394{\cdot}92 - 1071{\cdot}56 = 323{\cdot}36$ lbs., from which the advantage of using the force a_0 in the oblique direction becomes manifest.

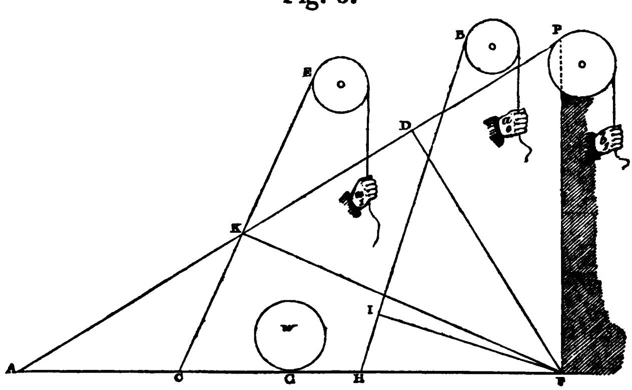
In the preceding process we have taken the value of $d_0 = 18{\cdot}6858836$ feet, being equal to the difference between the whole length of the lever F A and A H; that is, $\text{F H} = 56 - 37{\cdot}3141164 = 18{\cdot}6858836$; but the same thing may be determined directly from the data by equation (Q), on the supposition that the angle of direction when the force a_0 acts obliquely is known; that is, the value of the force b_0 corresponding to the vertical action of the force a_0 can be determined from the oblique action of a_0 without any previous calculation, by the following formula:

$$b_0 = \frac{\text{cosec. } \phi_n}{d_n} \left\{ \delta w - \frac{a_0 (d_0 \tan. \phi_0 \cot. \phi_n - d_n)}{\tan. \phi_0 \cot. \phi_n - 1} \right\}.$$

An equation which is very easily reduced by substituting the respective numbers as indicated by the combinations.

Let us now suppose that another subsidiary force as a_1 (fig. 6)

Fig. 6.



is added to the system, applied at the point C, and acting

in the direction C E. Then, since the weight of the lever remains the same, as well as the distance from the fulcrum of the point in which it is conceived to be concentrated, it will constantly require the same amount of power to effect the equilibrium, whatever may be the number of forces employed, in whatsoever manner they may be applied, and in whatever directions they may act.

Therefore, in the present case, with two subsidiary forces a_0 and a_1 acting in the directions H B and C E, and the third force b_1 acting in the direction A P, it will require the conjoint effects of all the three to maintain the equilibrium; that is, it will require the effect of the force a_0 in the direction H B, together with the effect of a_1 in the direction C E, and that of b_1 in the direction A P.

Now, by the notation already established, we have the angle F H B $= \phi_0$ and the distance F H $= d_0$, while the angle F A P $= \phi_n$ and the distance F A $= d_n$. In like manner let the angle F C E be denoted by ϕ_1, and the distance F C by d_1; then, if from the fulcrum F the perpendiculars F I, F K, and F D be respectively drawn, and each of them multiplied by the magnitude of its corresponding force, the equation of *equilibrium* arising from their conjoint effects becomes

$$a_0 d_0 \sin. \phi_0 + a_1 d_1 \sin. \phi_1 + b_1 d_n \sin. \phi_n = \delta w. \qquad \text{(R)}$$

Therefore, by reducing this equation in respect of b_1 we obtain

$$b_1 = \frac{(\delta w - a_0 d_0 \sin. \phi_0 - a_1 d_1 \sin. \phi_1) \operatorname{cosec.} \phi_n}{d_n}. \qquad \text{(S)}$$

If the equations (K) and (S) be compared with each other, it will be seen that the former exceeds the latter by the quantity

$$\frac{a_1 d_1 \sin.\,\phi_1 \operatorname{cosec.}\phi_n}{d_n},$$

for by subtraction we have

$$b_0 - b_1 = \frac{(\delta w - a_0 d_0 \sin.\,\phi_0) \operatorname{cosec.}\phi_n}{d_n} - \frac{(\delta w - a_0 d_0 \sin.\,\phi_0 - a_1 d_1 \sin.\,\phi_1) \operatorname{cosec.}\phi_n}{d_n} = \frac{a_1 d_1 \sin.\,\phi_1 \operatorname{cosec.}\phi_n}{d_n}.$$

Consequently, the tension on the cord or chain A P, when there are two subsidiary forces a_0 and a_1, is less than when there is only one subsidiary force, as a_0. The cord in the latter case may therefore be made smaller than it is in the former, for since there is less strain there is no necessity for so much strength.

If, instead of supposing the cords, to which the subsidiary forces are attached, to be brought over pulleys and stretched according to the intensity of their respective forces, let them be simply attached to the primary cord or chain A P, as represented in figs. 7 and 8 following; then, if they be stretched to the same degree by means of the load which they are employed to support, the nature of the action will be precisely the same as indicated in figs. 2 and 6, and the whole strain will be thrown upon the point of suspension at P, in the same

Fig. 7.

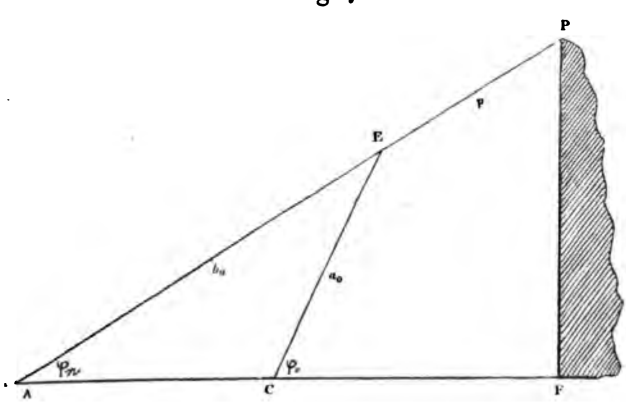
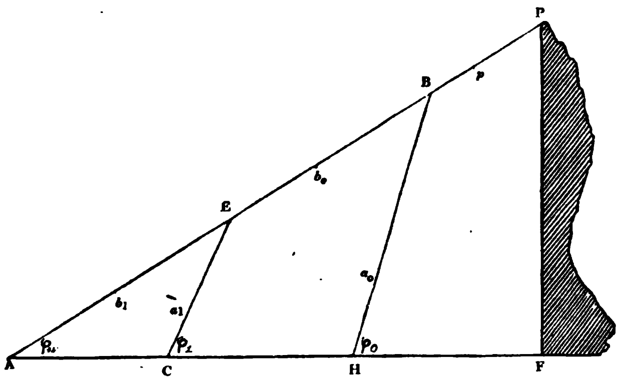


Fig. 8.



manner as if it were transmitted through the suspending bars of a bridge. From these two cases of subsidiary forces, as compared with each other, and also with the case of a single force, the law by which the tension on the cord or chain decreases becomes manifest; it is therefore unnecessary to pursue the investigation further, for by merely extending the notation the successive terms of the series can be obtained from each other by simple induction.

We shall give a tablet of the terms as far as seven subsidiary forces, which we presume will be amply sufficient to enable the attentive reader to detect the law by which the strain diminishes; and this law being once discovered, its application to the construction of suspension bridges becomes a matter of obvious utility. The series of equations denoting the state of equilibrium are as follow:

Force in the direction of the cord or chain . . . . $p\,d_n \sin.\phi_n = \delta w$

Ditto with one subsidiary force . . . . $a_0 d_0 \sin.\phi_0 + b_0 d_n \sin.\phi_n = \delta w$

Ditto with two subsidiary forces . . . . $a_0 d_0 \sin.\phi_0 + a_1 d_1 \sin.\phi_1 + b_1 d_n \sin.\phi_n = \delta w$

Ditto with three subsidiary forces . . . . $a_0 d_0 \sin.\phi_0 + a_1 d_1 \sin.\phi_1 + a_2 d_2 \sin.\phi_2 + b_2 d_n \sin.\phi_n = \delta w$

Ditto with four subsidiary forces . . . . . $a_0 d_0 \sin.\phi_0 + a_1 d_1 \sin.\phi_1 + a_2 d_2 \sin.\phi_2 + a_3 d_3 \sin.\phi_3 + b_3 d_n \sin.\phi_n = \delta w$

Ditto with five subsidiary forces . . . $a_0 d_0 \sin.\phi_0 + a_1 d_1 \sin.\phi_1 + a_2 d_2 \sin.\phi_2 + a_3 d_3 \sin.\phi_3 + a_4 d_4 \sin.\phi_4 + b_4 d_n \sin.\phi_n = \delta w$

Ditto with six subsidiary forces . $a_0 d_0 \sin.\phi_0 + a_1 d_1 \sin.\phi_1 + a_2 d_2 \sin.\phi_2 + a_3 d_3 \sin.\phi_3 + a_4 d_4 \sin.\phi_4 + a_5 d_5 \sin.\phi_5 + b_5 d_n \sin.\phi_n = \delta w$

Ditto with seven subsidiary forces $a_0 d_0 \sin.\phi_0 + a_1 d_1 \sin.\phi_1 + a_2 d_2 \sin.\phi_2 + a_3 d_3 \sin.\phi_3 + a_4 d_4 \sin.\phi_4 + a_5 d_5 \sin.\phi_5 + a_6 d_6 \sin.\phi_6 + b_6 d_n \sin.\phi_n = \delta w$

If we transpose this class of equations in such a manner that the terms involving the quantity b with its several characteristics may stand alone on one side, the effects of the forces on the successive portions of the cord A P, estimated in order from the point of suspension, will become obvious. The transposed equations are as below, having 7 instead of n, that being its numerical value for a system of eight forces.

$$p\,d_7 \sin.\,\phi_7 = \delta\,w$$

$$b_0 d_7 \sin.\,\phi_7 = \delta\,w - a_0 d_0 \sin.\,\phi_0$$

$$b_1 d_7 \sin.\,\phi_7 = \delta\,w - \{a_0 d_0 \sin.\,\phi_0 + a_1 d_1 \sin.\,\phi_1\}$$

$$b_2 d_7 \sin.\,\phi_7 = \delta\,w - \{a_0 d_0 \sin.\,\phi_0 + a_1 d_1 \sin.\,\phi_1 + a_2 d_2 \sin.\,\phi_2\}$$

$$b_3 d_7 \sin.\,\phi_7 = \delta\,w - \{a_0 d_0 \sin.\,\phi_0 + a_1 d_1 \sin.\,\phi_1 + a_2 d_2 \sin.\,\phi_2 + a_3 d_3 \sin.\,\phi_3\}$$

$$b_4 d_7 \sin.\,\phi_7 = \delta\,w - \{a_0 d_0 \sin.\,\phi_0 + a_1 d_1 \sin.\,\phi_1 + a_2 d_2 \sin.\,\phi_2 + a_3 d_3 \sin.\,\phi_3 + a_4 d_4 \sin.\,\phi_4\}$$

$$b_5 d_7 \sin.\,\phi_7 = \delta\,w - \{a_0 d_0 \sin.\,\phi_0 + a_1 d_1 \sin.\,\phi_1 + a_2 d_2 \sin.\,\phi_2 + a_3 d_3 \sin.\,\phi_3 + a_4 d_4 \sin.\,\phi_4 + a_5 d_5 \sin.\,\phi_5\}$$

$$b_6 d_7 \sin.\,\phi_7 = \delta\,w - \{a_0 d_0 \sin.\,\phi_0 + a_1 d_1 \sin.\,\phi_1 + a_2 d_2 \sin.\,\phi_2 + a_3 d_3 \sin.\,\phi_3 + a_4 d_4 \sin.\,\phi_4 + a_5 d_5 \sin.\,\phi_5 + a_6 d_6 \sin.\,\phi_6\}$$

The foregoing may be considered as a mathematical illustration of the principle upon which the tapering of the chain depends; but the subject is still involved in difficulty as regards the distribution of the forces, and the position of the oblique suspending rods; for here there is no conditional equation to direct us, and on this account the successful application of the principle to practice, must in a great measure depend upon the sagacity and skill of the engineer by whom the fabric is raised.'

In our humble opinion, a system of calculation may be instituted upon the principle of making the angles of direction vary by a constant second difference, at equidistant points along the platform, the limiting angles

being adapted to the circumstances of construction, which must of course be fixed upon before any calculation whatever can be made.

Another condition in the system of calculation might be, to have the effects of all the suspending forces equal; we do not mean the absolute magnitude of the forces, but the effects as referred to their particular directions, when compared with the state of equilibrium; this would manifestly give a series of equal differences for the tensions on the chain A P.

If the several equations above given be successively subtracted from one another, the series of corresponding remainders will stand as below, viz.:—

$$\left.\begin{array}{l}
(p_0 - b_7)\, d_7 \sin.\phi_7 = a_0 d_0 \sin.\phi_0 \\
(b_0 - b_1)\, d_7 \sin.\phi_7 = a_1 d_1 \sin.\phi_1 \\
(b_1 - b_2)\, d_7 \sin.\phi_7 = a_2 d_2 \sin.\phi_2 \\
(b_2 - b_3)\, d_7 \sin.\phi_7 = a_3 d_3 \sin.\phi_3 \\
(b_3 - b_4)\, d_7 \sin.\phi_7 = a_4 d_4 \sin.\phi_4 \\
(b_4 - b_5)\, d_7 \sin.\phi_7 = a_5 d_5 \sin.\phi_5 \\
(b_5 - b_6)\, d_7 \sin.\phi_7 = a_6 d_6 \sin.\phi_6
\end{array}\right\} \quad (T)$$

And if these equations be severally divided by the constant quantity $d_7 \sin.\phi_7$ the actual values of the differences will be as follows:—

$$p - b_0 = \frac{a_0 d_0 \sin.\phi_0 \operatorname{cosec.}\phi_7}{d_7}$$

$$b_0 - b_1 = \frac{a_1 d_1 \sin.\phi_1 \operatorname{cosec.}\phi_7}{d_7}$$

$$b_1 - b_2 = \frac{a_2 d_2 \sin.\phi_2 \operatorname{cosec.}\phi_7}{d_7}$$

$$b_2 - b_3 = \frac{a_3\, d_3 \sin. \phi_3 \operatorname{cosec.} \phi_7}{d_7} \qquad \text{(U)}$$

$$b_3 - b_4 = \frac{a_4\, d_4 \sin. \phi_4 \operatorname{cosec.} \phi_7}{d_7}$$

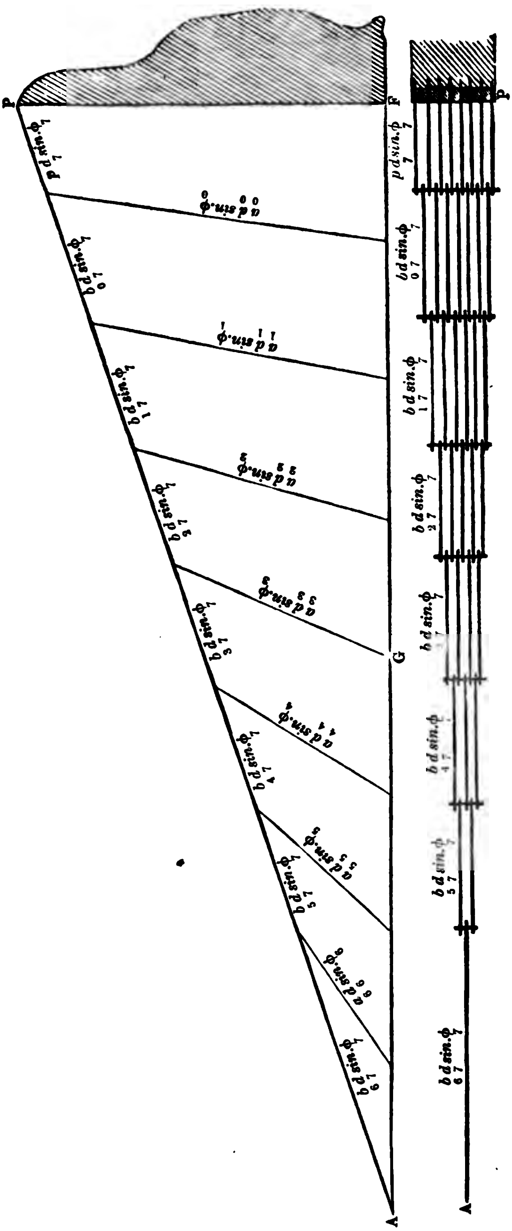
$$b_4 - b_5 = \frac{a_5\, d_5 \sin. \phi_5 \operatorname{cosec.} \phi_7}{d_7}$$

$$b_5 - b_6 = \frac{a_6\, d_6 \sin. \phi_6 \operatorname{cosec.} \phi_7}{d_7}$$

Consequently, the difference of the strains upon the first and last portions of the cord becomes

$$p - b_6 = \left\{ \frac{a_0 d_0 \sin. \phi_0 + a_1 d_1 \sin. \phi_1 + a_2 d_2 \sin. \phi_2 + a_3 d_3 \sin. \phi_3 + a_4 d_4 \sin. \phi_4 + a_5 d_5 \sin. \phi_5 + a_6 d_6 \sin. \phi_6}{d_7} \right\} \operatorname{cosec.} \phi_7$$

The following drawing will represent the several quantities involved in the equations, and the tapering plan of the chain will indicate the manner in which the strain diminishes by the application of the subsidiary forces.



ELEVATION.

PLAN OF THE CHAIN.

The following numerical example may be useful in showing how a calculation is to be instituted, supposing that the conditions previously mentioned are admitted as being sufficient to guide the operations of practice in a case of actual construction.

EXAMPLE.—Let A F in the preceding figure be equal in length to 120 feet, and suppose it to be a straight inflexible lever entirely divested of weight and moveable about the fulcrum at F. Let the distance A F be divided into eight equal parts of 15 feet each, and at the several points of division let forces be applied, whose directions make with the horizon angles of 81, 78, 73, 66, 57, 46, 33, and 18 degrees respectively, estimated in order from the fulcrum. Now, suppose a load of 250 tons to be applied at G, the middle of the lever, and to be kept in equilibrio by the simultaneous action of all the forces; it is required to determine the strains upon the different portions of the hypothenuse A P, admitting the several oblique forces acting in their respective directions to produce the same or an equal effect.

Since the load which balances the accumulated effect of all the forces is 250 tons applied at a distance of 60 feet from the fulcrum, the conditions of equilibrium as referred to the property of the lever require that

$$15\,a_0 \sin. 81^\circ + 30\,a_1 \sin. 78^\circ + 45\,a_2 \sin. 73^\circ + 60\,a_3 \sin. 66^\circ + 75\,a_4 \sin. 57^\circ + 90\,a_5 \sin. 46^\circ$$
$$+ 105\,a_6 \sin. 33^\circ + 120\,b_7 \sin. 18^\circ = 250 \times 60 = 15000.$$

Now, the eighth part of this, or 1875 tons, is the effect of each of the forces in producing the equilibrium; consequently the actual magnitude of each of the forces can easily be found as follows:—

Beginning with the oblique force which is nearest to the fulcrum, and proceeding in order to the most remote, the actual magnitudes of the forces to produce equal effects, accord-

ing to the obliquity of their directions, will be respectively as below:—

$$a_0 = \frac{p\, d_7 \sin. \phi_7}{8\, d_0 \sin. \phi_0} = \frac{1875}{15 \sin. 81^\circ} = 126{\cdot}5581 \text{ tons},$$

$$a_1 = \frac{p\, d_7 \sin. \phi_7}{8\, d_1 \sin. \phi_1} = \frac{1875}{30 \sin. 78^\circ} = 63{\cdot}8963 \text{ tons},$$

$$a_2 = \frac{p\, d_7 \sin. \phi_7}{8\, d_2 \sin. \phi_2} = \frac{1875}{45 \sin. 73^\circ} = 43{\cdot}5705 \text{ tons},$$

$$a_3 = \frac{p\, d_7 \sin. \phi_7}{8\, d_3 \sin. \phi_3} = \frac{1875}{60 \sin. 66^\circ} = 34{\cdot}2074 \text{ tons}, \qquad \text{(V)}$$

$$a_4 = \frac{p\, d_7 \sin. \phi_7}{8\, d_4 \sin. \phi_4} = \frac{1875}{75 \sin. 57^\circ} = 29{\cdot}8091 \text{ tons},$$

$$a_5 = \frac{p\, d_7 \sin. \phi_7}{8\, d_5 \sin. \phi_5} = \frac{1875}{90 \sin. 46^\circ} = 28{\cdot}9617 \text{ tons},$$

$$a_6 = \frac{p\, d_7 \sin. \phi_7}{8\, d_6 \sin. \phi_6} = \frac{1875}{105 \sin. 33^\circ} = 32{\cdot}7871 \text{ tons}.$$

These are the respective magnitudes of the subsidiary forces branching off from the primitive direction A P; and the corresponding magnitudes of the forces acting on the first and last portions of the hypothenuse or chain are as follows, viz.:—

$$p = \frac{\delta\, w}{d_7 \sin. \phi_7} = \frac{15000}{120 \sin. 18^\circ} = 404{\cdot}5086 \text{ tons},$$

$$b_6 = \frac{\delta\, w}{8\, d_7 \sin. \phi_7} = \frac{1875}{120 \sin. 18^\circ} = 50{\cdot}5636 \text{ tons}.$$

Having thus determined the magnitudes of all the subsidiary forces $a_0, a_1, a_2, a_3, a_4, a_5$ and a_6 together with the magnitudes of the forces which act on the first and last portions of the hypo-

thenuse or chain, it is easy to determine the magnitudes of the forces which act on all the intermediate portions; for, since by the conditions of construction, the effects of all the oblique forces are the same, it is obvious that all the equations in class (U) must be equal to the same constant quantity; that is, the forces acting on the different portions of the hypothenuse or chain, diminish in a regular progression from the highest to the lowest point, and the constant difference may be found from any one of the equations of the class just specified; or it may be found by taking the difference between the forces acting on the first and last portions, and dividing that difference by the number of terms diminished by unity. Now the force on the first portion is 404·5086, and that on the last is 50·5636 tons, and the number of terms or forces is 8; consequently we have $(404{\cdot}5086-50{\cdot}5636)\div 7=50{\cdot}5636$, which is the constant difference sought.

Therefore, if from the magnitude of the force which individually balances the weight, the constant difference be successively subtracted, the remainders will indicate the diminishing forces acting on the hypothenuse or chain; thus we have

$$b_0 = 404{\cdot}5086 - 50{\cdot}5636 = 353{\cdot}9450 \text{ tons.}$$

$$b_1 = 353{\cdot}9450 - 50{\cdot}5636 = 303{\cdot}3814 \quad ,,$$

$$b_2 = 303{\cdot}3814 - 50{\cdot}5636 = 252{\cdot}8178 \quad ,,$$

$$b_3 = 252{\cdot}8178 - 50{\cdot}5636 = 202{\cdot}2542 \quad ,,$$

$$b_4 = 202{\cdot}2542 - 50{\cdot}5636 = 151{\cdot}6906 \quad ,,$$

$$b_5 = 151{\cdot}6906 - 50{\cdot}5636 = 101{\cdot}1270 \quad ,,$$

$$b_6 = 101{\cdot}1270 - 50{\cdot}5636 = 50{\cdot}5634 \quad ,,$$

These forces, however, may also be deduced directly from the formula (K), instead of finding them as above from each other by continued subtraction; and since it may be necessary

on certain occasions to determine them in this way, it will be useful to show in what manner the reduction is to be effected.

By examining the several equations it will readily appear that $d_7 \sin. \phi_7$ in each of them is constant, and is equivalent to 120 sin. 18° = 37·08206; and the several portions of the entire effect which belong to the respective forces b_0, b_1, b_2, b_3, b_4, b_5, and b_6 are 13125, 11250, 9375, 7500, 5625, 3750, and 1875; so that we have

$$b_0 = \frac{13125}{37\cdot08206} = 353\cdot9450 \text{ tons.}$$

$$b_1 = \frac{11250}{37\cdot08206} = 303\cdot3814 \quad ,,$$

$$b_2 = \frac{9375}{37\cdot08206} = 252\cdot8178 \quad ,,$$

$$b_3 = \frac{7500}{37\cdot08206} = 202\cdot2542 \quad ,,$$

$$b_4 = \frac{5625}{37\cdot08206} = 151\cdot6906 \quad ,,$$

$$b_5 = \frac{3750}{37\cdot08206} = 101\cdot1270 \quad ,,$$

$$b_6 = \frac{1875}{37\cdot08206} = 50\cdot5635 \quad ,,$$

If we suppose all the values of ϕ but the last, or ϕ_7, to be equal to 90° or a right angle whose sine is unity; that is, if we suppose the subsidiary forces to act in directions perpendiçular to the platform or lever, then it is obvious that the actual values of those subsidiary forces will be less than when they act in oblique directions; and for this reason, the several forces which act in the direction of the hypothenuse must be proportionally augmented in order to produce the same effect at the point of suspension; hence the advantage of causing the subsidiary forces to act obliquely; and the same principle applies to the oblique suspending rods in Mr. Dredge's method of constructing bridges.

CONTRACT

FOR ERECTING

A SUSPENSION BRIDGE UPON MR. DREDGE'S PRINCIPLE AT BALLOCH FERRY, LOCH LOMOND.

It is contracted and agreed between the parties underwritten, viz., Sir James Colquhoun, of Luss, Bart., in the county of Dunbarton, North Britain, on the first part; and James Dredge, of Bath, in the county of Somerset, England, engineer, &c., as principal; and William Gibbons, of Bath aforesaid, maltster, &c., as guarantee, cautioner, and surety on the second part, in manner following: that is to say,—Whereas the said Sir James Colquhoun has resolved to erect a suspension bridge at Balloch Ferry, in the parish of Bonhill, Dunbartonshire, for the transit of foot passengers, horses, carriages, and cattle, conform to plans, elevation, and measurement, designed, drawn, and subscribed by the said James Dredge upon the principle invented by him, which plans consist of Nos. 1, 2, 3, and 4 different parts, which are all subscribed by the said parties as relative to these presents. Therefore, and in consideration of the price hereinafter mentioned, the said James Dredge has bound and obliged himself, and by these presents binds and obliges himself, and his heirs, executors, and representatives whomsoever, to construct and finish the said bridge, of sufficient strength, and in a good and workmanlike manner, conform to the plans, elevation, and measurement above mentioned; subject always to such alterations as shall appear to the said James Dredge to

be rendered necessary for the proper strength and durability of said erection.

And the said James Dredge obliges himself, at his own proper costs and charges, to provide the whole requisite working drawings and sections, models for masonry and iron-work, and also to furnish all iron rails and timber, and all other materials whatsoever (masonry materials excepted), which shall be necessary and fit to be used in or about the said bridge, and to superintend the erection of the same, and of the whole mason-work therewith connected.

And more particularly the said James Dredge binds and obliges himself, and his foresaids, to erect and finish the said bridge in manner following, viz.: of twenty feet width, and two hundred feet in length from the centre of one tower to the centre of the other, and of not less than seventeen feet and one half foot in height above the level of the River Leven at high flood, or at least as high from the surface of the river as the Bonhill Bridge, with an opening of forty feet outside of each tower. The platform of the bridge to be three inches thick, of larch timber, and the other parts of good iron and good workmanship, with iron gates, and the whole of the iron-work to be once well painted; and in general, without prejudice to the particulars above specified, the said James Dredge binds and obliges himself, and his foresaids, to construct and finish the said bridge and openings conform to the foresaid plans, elevation, and measurement, with such alterations as may be rendered necessary as aforesaid; and that the whole of the said bridge shall be executed in the best and strongest manner, and of the best materials.

And the said second parties hereby warrant that the said bridge, when completed, shall be capable of sustaining in transit double the weight of any load, whether of cattle, carriages, or otherwise, that it may ever be fairly exposed to. And further, the said James Dredge binds and obliges himself, and his foresaids, to proceed to the execution of the work above mentioned, and to carry on the same without any intermission, and completely finish the whole within two calendar months immediately after the completion of the towers of the said bridge. Providing hereby and declaring, that if the said

James Dredge shall fail so to complete the said bridge within said period of two months, he shall be bound to allow to the said Sir James Colquhoun, and the latter shall be entitled to retain from the price after specified, an abatement of five per cent. upon the amount which the said Sir James Colquhoun shall have advanced for masonry on the said bridge, and that during the time the said bridge shall remain unfinished after the expiring of the foresaid two months.

And it is hereby agreed that if the fencing off of a foot-path upon said bridge shall be dispensed with by the said Sir James Colquhoun, he shall in that case be allowed a deduction of twenty pounds from the stipulated price of said bridge hereinafter mentioned; and in the event of his requiring only two gates on said bridge in place of three shown on said plan, he shall receive a further deduction on that account of five pounds sterling. And further, if any other part of the said plans shall be dispensed with by the said Sir James Colquhoun, or not executed by the said James Dredge, the expense of such part or parts shall also be deducted from the price after mentioned. And if any of the conditions or particulars above specified shall be altered or executed contrary to the foresaid plans, these shall not be construed into an abandonment of the said plans, but the same shall be completed, and the present contract shall be effectual notwithstanding such alterations.

And further, the said second parties, viz., the said James Dredge, as principal, and the said William Gibbons, as cautioner, guarantee, and surety for him as aforesaid, bind and oblige themselves, conjunctly and severally, and their foresaids, to warrant, uphold, and maintain the said bridge and towers thereof in good and sufficient condition for the space of five years after completion thereof, excepting from this obligation any injury or damage which shall arise thereto by lightning, earthquake, civil commotion, or malicious damage, or not fairly within the compass of wear and tear by ordinary use of such bridge. And if during the said period of five years any parts of said towers or bridge be discovered to be insufficient or requiring repair (not rendered necessary from the causes aforesaid), then the second parties shall be bound and

obliged, as they do hereby bind and oblige themselves and their foresaids, to repair such parts and make them sufficient at their own expense immediately on the insufficiency being discovered. For which causes, and on the other part, the said Sir James Colquhoun binds and obliges himself, his heirs and successors, immediately on such bridge being completed and proved to be sufficiently strong in all its parts as after mentioned, which proof shall be ascertained within one calendar month after its completion, to make payment to the said James Dredge, or his heirs, executors, or assigns, of the sum of one thousand five hundred pounds sterling, as the agreed price of the said bridge to be constructed by the said James Dredge in terms hereof, including therein the whole of his charges for engineering, travelling, and all other charges and expenses competent to him for said work, excepting such reasonable charge for superintending the erection of the said towers, either by himself or by a qualified person appointed by him, as may be considered fair between the parties, or as shall be fixed by reference to a respectable tradesman resident in Dunbartonshire, in case they do not agree. It being here contracted and agreed that the said Sir James Colquhoun shall, when required by the said James Dredge, furnish the timber requisite for said bridge, and be entitled to retain out of the said price of one thousand five hundred pounds the current price of such timber so furnished, besides the expense of dragging, conveyance, and whole workmanship thereof; and on the completion of said bridge, the price of said timber and the said expenses attending the same (which last shall be ascertained and proved by the account and statement of the wood forester on the estate of Luss) shall be deducted from the said stipulated sum of one thousand five hundred pounds; and the balance thereof, after such deduction, shall then be immediately payable to the said James Dredge and his foresaids. Providing always that the said bridge, including mason-work, shall have been previously duly tested and proved, within the aforesaid period of one calendar month, to be of the requisite strength, and completed in all respects according to agreement, and in such a manner as to afford to the public a safe com-

munication across the river Leven. But it is hereby expressly provided and declared that, if between the completion of the said bridge and the expiration of said five years, the bridge, including towers, shall not have fairly borne its work, and be thus proved unable or insufficient to answer the purposes for which the bridge is intended, the said James Dredge, as principal, and the said William Gibbons, as guarantee, cautioner, and surety foresaid, shall be bound and obliged, as they hereby bind and oblige themselves and their aforesaids, conjunctly and severally, to repair and make the bridge substantial in every respect, or to make repayment to the said Sir James Colquhoun, his heirs and assigns, not only of the foresaid sum of one thousand five hundred pounds sterling, but also of the further sum of two hundred pounds as and for a moiety of the outlay and expense disbursed by the said Sir James Colquhoun in the erection of the mason-work connected with the said bridge; and on such repayment, the said James Dredge and his foresaids shall be fully entitled to the whole of the iron and wood-work of the said bridge as his own property, and all the materials excepting the stone and mason-work thereof. And it is further hereby provided that, in the event of any difference arising with respect to the true meaning of the present contract or the execution of any part of the work hereby contracted for, the parties hereby submit the same to the final determination of William Steele, Esq., Advocate, Sheriff Substitute of Dunbartonshire, and failing him, to the Sheriff Substitute of the said shire for the time being, as sole arbiter, and oblige themselves and their foresaids to abide by and fulfil any decision which he shall pronounce on the matters hereby submitted to him. And both parties bind and oblige themselves and their foresaids to implement and perform their respective parts of the premises to each other under the penalty of five hundred pounds sterling, to be paid by the party failing to the party observing the contract or willing to do so, besides performance. And both parties consent to the registration hereof in the Books of Council and Session Sheriff Court of Dunbartonshire or other Judges' Books competent for preservation; and that letters of horning, or six days' charge, and all other

execution needful, may hereon pass on a decree to be interponed hereto, and for that purpose constitute

Procurators; in witness whereof these presents, written upon stamped paper by James Mackibbin, Clerk to Robert Grieve, Writer, Dunbarton, are subscribed,

(L. S.)

(L. S.)

(L. S.)

SPECIFICATION OF THE QUANTITIES OF MATERIAL

USED IN THE

SUSPENSION BRIDGE AT BALLOCH FERRY,

DUNBARTONSHIRE.

THE PLATE is an isometrical projection of a suspension bridge now in course of erection across the Leven at Balloch Ferry in Dunbartonshire, for Sir James Colquhoun, of Luss, Bart., after designs and under the superintendence of Mr. Dredge, of Bath.

The whole extent of the suspended road-way is 292 feet; but as a space is left on each side as an easy access to and from the towing-paths, the distance from the centre of one pier to the centre of the other is reduced to 200 feet, which must be considered as the span of the bridge.

Each pier for supporting the chains consists of two octagonal towers upon which the chains rest, and which are connected together by springing a light arch across the road-way. These towers taper from 15 feet 2 inches by 9 feet 2 inches at the base, to 9 feet by 3 feet at the top, the longest dimension being in the direction of the bridge. The entire height of the towers is as under, viz.

	feet.
From bed of river to water-mark	3
,, water-mark to upper surface of road-way . .	16
,, road-way to tops of towers	21
	40 feet.

The distance from the centre of one octagonal tower to the centre of the other across the road-way is 20 feet, equal to the width of the bridge. The towers and the key-stone of the arch are built of a reddish freestone from Bonhill; the stringing courses at the road-way, springing of the arch, and the offsets at the top of the tower, of a white stone in the neighbourhood.

Upon the tops of the towers are large cast-iron plates, on which the chains rest.

The principal chains for supporting the structure are formed of $\frac{7}{8}$ round bars, laid side by side to the number of 13 upon the tops of the towers, and successively dropping one at each joint until it arrives at the centre, where that part of the chain is reduced to a single bar. The links upon the towers are 6 feet long, but after springing from thence they are increased to 9 feet.

1. The weight of each link, 6 feet long, is 14·5 lbs.

The weight of iron on the towers is therefore
$14{\cdot}5\,(13\times4)$ . . . . $=$ 754

The weight of each link, 9 feet long, is 20 lbs.

The weight of iron is therefore $20\{8(12+2\times5{\cdot}5)+4\}=$ 12400

$(24\times4)=96$ connecting bolts increasing in arithmetical progression from 3 to 12 lbs. each,
$\frac{96}{2}(3+12)$ . . . . $=$ 720

Weight of iron in chains . . $=$ 13874 lbs.

The length of the oblique suspending rods, if radius remained constant, would be as the sec. of the angle which they respectively make with the horizon; but in consequence of the inclination of the chains towards the joints of connexion with the suspending rods, their length diminishes continually towards the centre of the bridge; and as the ascertaining the exact length of each rod separately would involve much complicated calculation, we will avoid it, and assume (which will be very near the truth) the average length of the bars to be 11 feet 9 inches, (excepting those immediately proceeding from

the joints on the towers:) the two bars from each joint of the chains being $\frac{3}{4}$ of an inch in diameter, we have, for the weight of the whole of the $\frac{3}{4}$ bars,

2.	*1·47 (88 × 2 × 11·75)	=	3039·96
	16 oblique rods from the towers $\frac{7}{8}$ diameter, averaging 26·6 feet in length:		
	*2 (16 × 26·6)	=	851·2
	136 pieces $\frac{7}{8}$ rod, each 2 feet long, welded to the end of the oblique rods:		
	2 (136 × 2)	=	544·
	136 nuts, each ·25 lbs.		
	·25 × 136	=	34·
	56 pieces 1 inch in diameter, 2 feet long, in retaining rods:		
	2·61 (56 × 2)	=	292·32
	56 nuts, each ·333 lbs.		
	·333 × 56	=	18·64
	152 nuts and pins, each ·5 lbs.		
	·5 × 152	=	76·
	Total weight of oblique suspending rods	=	4856·12 lbs.

Proceeding onwards, we next arrive at the horizontal beams, or those running from end to end of the bridge; and there are here 2 of 190 feet long, and 4 of 45 feet long each, all 5 inches × $\frac{1}{2}$ inch, so that

3.	8·4 {(190 × 2) + (45 × 4)}	=	4704
	64 connecting plates, each 1 foot long:		
	8·4 × 64	=	537·6
	6 $\frac{5}{8}$ pins to each plate, each ·5 lbs.		
	·5 × 64 × 6	=	192
	Total weight of horizontal beams		5433·6 lbs.

We have now to examine the weight of the transverse beams, or those which run at right angles to and immediately support

* Where 1·47 and 2 are respectively the weight in lbs. of a $\frac{3}{4}$ and a $\frac{7}{8}$ rod 1 foot long.

the platform; they are each 20 feet long, 5 inches deep × $\frac{1}{2}$ an inch in thickness, and have at every 6 inches along their length, near the upper edge, holes of $\frac{11}{16}$ths of an inch drilled to admit of $\frac{5}{8}$ rods passing longitudinally through them from end to end of the bridge; there are 117 such beams in all:

4. $8{\cdot}4\ (20 \times 117)$ . . . . . . $= 19656$

Every third beam is trussed with suspension trusses, consisting of 5 rods, viz., $1 = \frac{7}{8}$ and $4 = \frac{5}{8}$, each 6·66 feet long:

$2\ (39 \times 6{\cdot}66)$ . . . $= 519{\cdot}48$

$1{\cdot}02\ (39 \times 4 \times 6{\cdot}66)$ . . $= 1059{\cdot}739$

$1579{\cdot}219$

$\frac{1579{\cdot}219 \times 8}{100}$ . . . $= 126{\cdot}337$

$1705{\cdot}556 = 1705{\cdot}55$

$21361{\cdot}55$

From which deduct for the quantity of iron not used in the holes,

$1{\cdot}23\left(\frac{40 \times 117}{24}\right)$ . . . . . $= 239{\cdot}85$

$21121{\cdot}70$

The $\frac{5}{8}$ rods to which we before alluded run 6 inches apart through the transverse beams from end to end, and are placed there for the purpose of combining the whole mass together, and causing a more equable pressure upon the beams which support the road-way; there are, therefore, 40 rods of 195 feet long each, and 80 of 45 feet, so that the whole weight will be

5. $1{\cdot}02\ \{(195 \times 40) + (80 \times 45)\} = 11628.$

There are also 28 bars, each 10 feet long, $1\frac{1}{8}$ in diameter, to be used as holdfasts in the ground for retaining the chains.

6. $3{\cdot}31\ (28 \times 10)$ . . . . . . $= 926{\cdot}8$

28 nuts, ·75 lb.

$\cdot 75 \times 28$ . . . . $= 21$

947·8 lbs.

And by adding the results together we shall have

1	. .	= 13874
2	. .	= 4856·12
3	. .	= 5433·6
4	. .	= 21121·7
5	. .	= 11628
6	. .	= 947·8

		tons.	cwt.	qr.	lbs.
Total	57861·22 =	25	16	2	13·22

the whole of the wrought-iron actively employed in the structure; to which, if we add that used in railing,

2 (400 × 2 × 4)	= 6400
and about 400 lbs. more as staples, &c. . .	400
	6800 lbs.

tons.	cwt.	qr.	lbs.
3	0	2	24
25	16	2	13·22
28	17	1	9·22

the total quantity of wrought-iron used in the bridge.

We have next to examine the cast-iron used; and beginning with the plates on which the chains rest,

1. Four plates, each 560 lbs. . = 2240

136 circular plates for fastening the oblique rods to the horizontal beam, each 3·5 lbs.

2. 136 × 3·5 . . . . . = 476

22 lbs. cast blocks to each truss, and there being 39 trusses:

22 × 39 . . . . . = 858

2 cast boxes for each beam for attaching it to the horizontal longitudinal beams, each 11 lbs.

11 × 2 × 117 . . . . = 2574

28 castings in retaining chains, each 5 lbs.

5 × 28 . . . . . = 140

	tons	cwt.	qr.	lbs.
6288 lbs. =	2	16	0	16

as the total quantity of cast-iron actively employed in the bridge; to which add the quantity used in the railing,

	tons	cwt.	qr.	lbs.
$28 \times 4 \times 35 = 3920 =$	1	15	0	0
	2	16	0	16
	4	11	0	16

The timber used in its construction is larch, grown on the estate of Sir James Colquhoun at Luss, and the quantity used is,

$292 \times 20 \times \cdot 25$	$= 1460$
2 beams of wood passing from end to end, 6×3 inches:	
$2\ (292 \times \cdot 5 \times \cdot 25)$	$= 73$
	1533 feet.

The whole of the material used is,

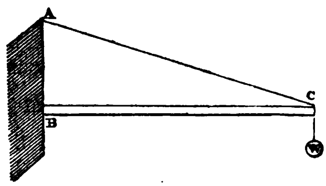
	tons	cwt.	qr.	lbs.
1st, Wrought-iron	28	17	1	9·22
2nd, Cast-iron	4	11	0	16
3rd, Timber	1533 cubic feet.			

The principle upon which this bridge is built is entirely new; and it has been doubted by many whether the arrangement of the bars in the chain and the adoption of the oblique suspending rods is a judicious application of iron in the construction of such bridges. The small quantity of material used in the one specified above, (which is intended as a permanent structure for general traffic,) is however, a sufficient proof; and we shall endeavour to show by mathematical reasoning, that it is more than equal to sustain any load which it is possible to place upon the surface of its road-way.

In calculating the force of a bridge of this description, the formulæ involved are necessarily very different from those used in estimating the common catenary; for here one half of the bridge balances the other, the structure being held in equilibrio entirely by the acting chains, and the centre part of the chain where it is reduced to a single link, may be severed or entirely removed without affecting the stability of the structure. It is therefore evident that such a bridge may be

referred to the projection of two brackets, and estimated accordingly.

Fig. 1.



Conceive the weight W to be supported by a cord A C at the extremity of a horizontal projection B C, and maintained there by a tension in the line A C, and compression in the line B C; then A B C being a right angled triangle by construction,

$$\sin. ACB : \text{rad.} :: AB : AC,$$

$$\text{or } AC = \frac{\text{rad.}}{\sin. ACB} . AB = AB. \ \text{cosec.} ACB$$

by the principles of projection.

$$AB : (AC) = AB. \text{cosec.} ACB :: W : W \frac{\text{cosec.} ACB . AB}{AB} =$$

W cosec. A C B.

Let $w = W$, the weight resting on the point C.

$\theta = \angle ACB$, contained by the horizontal line and the line A C.

$x =$ the tension in the line A C (caused by the weight W) in the direction C A.

$y =$ pressure or thrust in the direction C B.

Then, by substituting these quantities for the former proportion, we have

$$\sin. \theta : \text{rad.} :: w : x,$$

and by equating the products and reducing the equation, we get

$$x = \frac{\text{rad.}}{\sin. \theta} w = w \ \text{cosec.} \theta \quad . \quad . \quad . \quad . \quad . \quad (1)$$

And for the pressure or thrust in the direction C B, it is

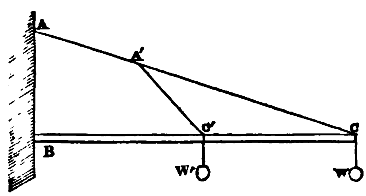
$$\sin. \theta : \cos. \theta :: w : y;$$

consequently by reduction, we have

$$y=\frac{\cos.\theta}{\sin.\theta}w=w\cot.\theta \quad \ldots\ldots \quad (2)$$

Which equation holds true for every value of the angle θ; but there is the same strain in every part of the line A C, and if such were to remain, there would be no occasion to increase the size of the chain towards the point A. But let us further suppose another weight W′ to be supported by a line from some intermediate point between B and C, and let that line be attached to some point A′ in the cord A C, as in fig. 2.

Fig. 2.



Allowing A C B to remain as before, and supposing the same quantities to retain the same values as in the former figure, and putting in this w', θ', x', y' respectively for the weight W′, the < A′ C′ B, the tension in the line A′ C′, and the pressure in the direction C′ B, we have, for the value of x' by equation (1),

$$x'=w'\operatorname{cosec}.\theta' \quad \ldots\ldots\ldots \quad (3)$$

The point A′ in the line A C is acted upon by two forces represented by the symbols x, x', the resultant of which forces in the direction towards A will be,

$$R=\sqrt{x^2+x'^2+2\,x\,x'\cos.(180^\circ+\theta-\theta')} \quad \ldots \quad (4)$$

The former values of x and x' being substituted, we get

$$R=\sqrt{w^2\operatorname{cosec}.^2\theta+w'^2\operatorname{cosec}.^2\theta'+2w\operatorname{cosec}.\theta\,.\,w'\cos.\theta'\cos.(180^\circ+\theta-\theta')}$$

And the difference will be represented by

$$D=\sqrt{w^2\operatorname{cosec}.^2\theta+w'^2\operatorname{cosec}.^2\theta'+2w\operatorname{cosec}.\theta\,.\,w'\cos.\theta'\cos.(180^\circ+\theta-\theta')} \\ -w\operatorname{cosec}.\theta,$$

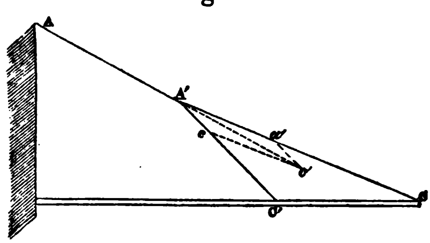
and in which proportion, the chains should increase in sectional area towards the base. Proceeding in like manner for another section, the resultant of the forces would be

$$R'' = \sqrt{(x''^2 + R^2 + 2\,R\,x''\cos.\,(180^\circ + \delta - \theta''))}^* \quad . \quad (5)$$

From which it appears, that each progressive section of the chain towards the point of suspension, has an increase of strain, equal to the resultant of the two forces acting at the further extremity of that section towards the extremity of the bracket; and from this it is evident, that the chain should taper in the same proportion; but it is equally clear, that the use of the oblique suspending rods is peremptorily insisted upon, otherwise the reduction of material towards the extremity of the bracket beyond a certain extent would be a proportionate reduction of power.

The following diagrams will graphically illustrate this:

Fig. 3.



Draw $a'\,c'$ (fig. 3) parallel to $A'\,C'$, produce $A\,A'$ onwards to the point c', and the triangle of forces $A'\,a'\,c'$ being complete, it only remains that the lines supporting the brackets should be increased in power proportionally to the length of the sides of the triangle to which they are respectively parallel.

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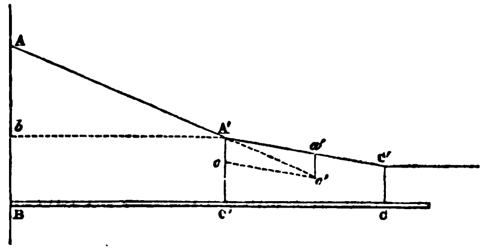
* Where x'' represents the value of the tension sustained in the angle or inclination θ'', R the resultant of the first two forces, and δ an angle of which R is equivalent to the cosec.

Fig. 4.*



Again (fig. 4), draw a' c' parallel to A′ C′, and continue A A′ to c'; then will the triangle of forces be complete, as in the former curve; but it is evident by the construction of the bracket, that the pressure of the horizontal force is not sustained by the road-way. Draw A′ b parallel to B C, then will A′ b be the measure of that horizontal force, or equal to the weight resting on A′ drawn into the cotangent of the angle A A′ b, which must be counteracted by a sufficient power in the direction A′ C′; and the weight of the iron, which would be equivalent to that power, would be oppressing the chain in the direction A′ A, as the cosecant of the angle A A′ b.

It is well known that the parameter of the catenary of a suspension bridge, is equivalent to half the weight of the structure drawn into the cotangent of the angle which a tangent to the curve at the point of suspension forms with the horizon, and the oblique strain is equal to half the weight drawn into the cosecant of the same angle.

Therefore, from what has been said, it is clear that a parallel section of iron, equivalent to the parameter of the catenary, may be taken away from one end of the curve to the other, and that not only the weight would be saved, but also another

* I would here presume that whilst speaking of a bracket supported by a catenary curve with vertical suspending rods, a tensile force must be acting from the point C′ equivalent to the parameter of that curve.

quantity of iron which was necessary to support it. These, then, are the advantages which arise from the use of the oblique suspending rod.

Example.

Suppose A B C (fig. 1) to be a bracket projecting from a wall with a weight of 364 lbs. attached at the point C, the angle A C B being 39°; required the tensile power upon the cord A C, and also the horizontal pressure in the direction B C?

By equation (1) we have,

$$x = w \text{ cosec. } \theta,$$

therefore,

$$364 \times 1{\cdot}589015 \quad . \; . \; . \; . \; . \quad = 578{\cdot}401 \text{ lbs.},$$

and by logarithms,

$$\begin{array}{lrl} \text{log. } 364 \quad . \; . & = & 2{\cdot}561101 \\ \text{log. cosec. } 39^\circ & = & 10{\cdot}201128 \\ \hline \text{nat. num.} & & 2{\cdot}762229 \quad = 578{\cdot}401 \text{ lbs.} \end{array}$$

As shown by equation (2),

$$y = w \text{ cot. } \theta,$$

hence

$$364 \times 1{\cdot}234897 \quad . \; . \; . \; . \; . \quad = 449{\cdot}502 \text{ lbs.},$$

by logarithms,

$$\begin{array}{lrl} \text{log. } 364 \quad . \; . & = & 2{\cdot}561101 \\ \text{log. cot. } 39^\circ \quad . & = & 10{\cdot}091631 \\ \hline \text{nat. num.} & & 2{\cdot}652732 \quad = 449{\cdot}502 \text{ lbs.} \end{array}$$

Again, let every thing remain as before; but suppose another weight equal to 364 lbs. were added to the point C′, as in fig. 2, and supported at an angle of 50°; required the increase of strength from A′ towards A, and also the horizontal pressure towards B?

By equation (3),

$$x' = w' \text{ cosec. } \theta';$$

consequently,

$$364 \times 1{\cdot}305408 \quad . \quad . \quad . \quad . \quad . \quad = 475{\cdot}16 \text{ lbs.}$$

$$\begin{array}{lrl} \text{log. } 364 \; . \; . \; . & = & 2{\cdot}561101 \\ \text{log. cosec. } 50^\circ \; . & = & 10{\cdot}115746 \\ \hline \text{nat. num.} & & 2{\cdot}676847 = 475{\cdot}16 \text{ lbs.} \end{array}$$

Therefore 475·16 lbs. is the tension in the direction C′ A′; but to ascertain the value of the tension from A′ to A, by referring to equation (4) we find,

$$R = \sqrt{x^2 + x'^2 + 2\,x\,x' \cos.\,(180^\circ + \theta - \theta')},$$

and by substituting the numerical values for the several quantities, we get

$$R = \sqrt{578{\cdot}401^2 + 475{\cdot}16^2 + 2 \times 578{\cdot}401 \times 475{\cdot}16 \times {\cdot}981627}$$

that is

$$578{\cdot}401^2 = 334547{\cdot}7168$$

$$475{\cdot}16^2 = 225777{\cdot}0256$$

$$2 \times 578{\cdot}401 \times 475{\cdot}16 \times 981627 = 539567{\cdot}0241$$

and

$$\sqrt{1099891{\cdot}7665} = 1048{\cdot}75 \text{ lbs.}$$

The required strength in that part of the cord A A′, and the difference between the part A A′ and A′ C is,

$$\begin{array}{r} 1048{\cdot}75 \\ 578{\cdot}401 \\ \hline 470{\cdot}349 \end{array}$$

And the sectional area of the one part should be to the sectional area of the other as

$$1 : 1{\cdot}8131.$$

For the magnitude of the horizontal force; that which is occasioned by the weight at the further extremity of the bracket has been shown to be = 449·502 lbs.; but the adding of another weight at C′ will cause an increase towards B, so that the aggregate of force from C′ to B is represented by

$$449{\cdot}502 + (364 \times {\cdot}839100) = 754{\cdot}934 \text{ lbs.}$$

And the difference of pressure between the part C′ B and C C′ is

$$754{\cdot}934 - 449{\cdot}502 = 305{\cdot}432.$$

Example 2.

Suppose a figure similar in every respect to fig. 4, and let a weight of 364 lbs. be attached to the horizontal line from the point C, and let A′ C′ form the same angle with the horizon that A′ C′ did in the former curve, viz., 39.° Then the tension in the direction A′ C′ will be

$$364 \times 1{\cdot}589015 = 578{\cdot}400 \text{ lbs.}$$

and the horizontal force, as before represented, is equal to

$$364 \times 1{\cdot}234897 = 449{\cdot}502 \text{ lbs.},$$

which must in this case be resisted through the cord in the direction from C; or, in short, as the parameter of the catenary; and therefore, the absolute tension in the direction C′ A is in consequence, equal to 1·589015 (364 + the weight of material requisite to sustain a pull of 449·502 lbs.), and which, in large spans, increases to such an enormous extent as to facilitate the destruction of the structure.

The throwing the thrust on the towers or abutments into the horizontal line is of advantage to the road-way, as rendering it more steady and not so liable to that oscillation to which suspension bridges are commonly exposed; and in practice, the sectional area of the beam to sustain the transverse action of the platform, would be more than sufficient under any circumstance, to overcome the horizontal action to which the structure may be exposed: for that reason, therefore, it would be useless to increase the beam towards B.

The preceding examples which refer to the bracket only are strictly arbitrary, and have been proposed for the purpose of showing in what manner the formulæ arising from the investigation are to be applied; but as a further illustration of the advantages to be derived from the adoption of the system just

discussed, we shall revert to the bridge across the River Leven for which the quantity of materials has already been ascertained. The whole quantity of material employed is apparently very small, and if compared with that which would be required on the old principle, might tend to create a doubt as regards the strength and stability of the structure. In order, therefore, to remove any suspicions that may be entertained on that point, we shall here endeavour to show that the material used, though small in quantity, is amply sufficient to sustain a much greater load, than under any possible circumstances can ever be brought upon the platform.

There are four points of suspension, and at each point there are twelve bars in the chains besides two oblique suspending rods, all of $\frac{7}{8}$ths of an inch in diameter; this gives a section of 8·4185 square inches at each tower or point of support, or, in all, 33·674 square inches of section. Now, since the absolute tensile power of a $\frac{7}{8}$th bar of malleable iron of a medium quality is 35000 lbs. very nearly, the absolute strength of the iron at the joints adjacent to the towers is 875 tons, for

$$(12=2)\times 4\times 35000=1960000 \text{ lbs.}=875 \text{ tons.}$$

In the next place, we have to ascertain the greatest tension, that under any circumstances can come upon the chains at the points of support, and for this purpose we shall suppose the weight of the platform, chains, diagonal rods, &c., between the towers to be 67200 lbs., and taking the greatest extraneous load to which a bridge ought ever to be exposed at 120 lbs. per square foot, we get

$$190\times 20\times 120=456000 \text{ lbs.}$$

the length of the platform being 190 feet; therefore we have

$$67200+456000=523200 \text{ lbs.}$$

for the total weight to be supported.

Now, the angle which the first series of bars would make with the platform, if continued downwards till they intersect it, is 24 degrees, and the natural cosecant of 24 degrees, is 2·458591; consequently, by equation 1 (page 55), we obtain for the effect of obliquity,

$$523200 \times 2{\cdot}458591 = 1286334{\cdot}8112 \text{ lbs., or } 574{\cdot}2566 \text{ tons.}$$

Let this be subtracted from the absolute power of the iron at the points of suspension, and we have

$$875 - 574{\cdot}2566 = 300{\cdot}7434 \text{ tons,}$$

which is the surplus strength in the chains after the platform is fully loaded; and every part of the structure, if properly tested, would be found equally sufficient for the intended purpose.

The preceding method of obtaining the surplus strength of the bridge is simple and easily applied, but it may be done otherwise as follows.

We have found that the entire weight to be supported in an extreme case is 523200 lbs., and we have moreover seen, that the ultimate strength of a single bar of the iron used, is 35000 lbs.; consequently, the number of bars required at the towers will be

$$523200 \div 35000 = 14{\cdot}948 \text{ bars.}$$

This, however, is on the supposition that the iron is strained vertically in the direction of its fibres, but this in the case of a bridge is contrary to fact, the bars being all placed in an oblique position with respect to the horizon, while the weight acts in a direction perpendicular to it; the power of the iron must therefore be reduced in the ratio of radius to the sine of the angle of obliquity; or, which is the same thing, the weight must be increased in the ratio of the cosecant to radius. Now the angle of obliquity is 24 degrees, and its cosecant 2·458591; hence we get

$$523200 \times 2{\cdot}458591 \div 35000 = 36{\cdot}75 \text{ bars;}$$

but there are 56 bars at the towers in the Balloch Bridge; consequently we have

$$56-36{\cdot}75=19{\cdot}25 \text{ bars},$$

or 4·8125 bars at each tower for the surplus strength, even in the case of an extreme load.